DIFFERENTIAL EQUATIONS

COHEN

DONALD L. WONG

Digitized by the Internet Archive
in 2022 with funding from
Kahle/Austin Foundation

https://archive.org/details/elementarytreati0000unse_o6o8

AN ELEMENTARY TREATISE

ON

DIFFERENTIAL EQUATIONS

SECOND EDITION, REVISED AND ENLARGED

BY

ABRAHAM COHEN, PH.D.
ASSOCIATE PROFESSOR OF MATHEMATICS
THE JOHNS HOPKINS UNIVERSITY

D. C. HEATH AND COMPANY
BOSTON NEW YORK CHICAGO
ATLANTA SAN FRANCISCO DALLAS
LONDON

AN ELEMENTARY TREATISE
ON
DIFFERENTIAL EQUATIONS

SECOND EDITION, COMPLETELY REVISED

BY

ABRAHAM COHEN, PH.D.
COLLEGIATE PROFESSOR OF MATHEMATICS
THE JOHNS HOPKINS UNIVERSITY

D. C. HEATH AND COMPANY
BOSTON NEW YORK CHICAGO
ATLANTA SAN FRANCISCO DALLAS
LONDON

COPYRIGHT, 1933
BY ABRAHAM COHEN

No part of the material covered by this copyright may be reproduced in any form without written permission of the publisher.

4G8

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

The following pages are the result of a course in Differential Equations which the author has given for some years to classes comprising students intending to pursue the study of Engineering or some other Physical Science, as well as those expecting to continue the study of Pure Mathematics or Mathematical Physics.

The primary object of this book is to make the student familiar with the principles and devices that will enable him to integrate most of the equations he is apt to come across. As much of the theory is given as is likely to be comprehensible to the student who has had a year's course in the Differential and Integral Calculus, and yet is sufficient to form a harmonizing setting for the numerous and otherwise apparently miscellaneous classes of equations, and the disconnected methods for solving them. It is intended to have the work sufficiently broad to make it a handy book of reference, without affecting its utility as a text-book. A number of footnotes and remarks have been put in, which, without breaking the continuity of the practical side of the subject, must prove of interest and value. Numerous historical and bibliographical references are also made.

* * * * * * * * * * * * *

The author has had in mind continually the necessity of systematizing the various classes of equations that can be solved by elementary means, and of minimizing the number of methods by which they can be solved. To enable the student to get a better general view of the subject, the summaries at the ends of the various chapters and the final general summary must prove of great value.

Numerous applications to problems in Geometry and the Physical Sciences have been introduced, both in the body of the text and in the form of exercises for the student.

Although a large number of the problems have been published before, many are new, and all have been chosen to bring out the various methods of the differential equations, and of the integral calculus as well. Many of the examples worked out in the text were chosen to recall some of the more important methods of the latter; for while the use of tables of integrals is recommended, the student should not feel absolutely dependent upon them. Most of the solutions have a simple form or an interesting interpretation. Great care has been taken to avoid typographical errors. The author shall be very glad to learn of any that still exist.

The method of undetermined coefficients for finding the particular integral in the case of linear equations with constant coefficients is believed to be presented here for the first time in its complete form.

The subject of Partial Differential Equations is so vast that it was decided to present only a few topics, which, in all probability, will suffice for the needs of the students for whom this book is intended.

It was only after considerable thought that the author refrained from adding a chapter on the Lie Theory. It is hoped to present that important branch of the subject in a separate volume.

In conclusion the author takes great pleasure in expressing his appreciation of the valuable suggestions made by Professor F. S. Woods of the Massachusetts Institute of Technology, as well as of those by Professor L. G. Weld of the University of Iowa and Professor E. J. Townsend of the University of Illinois.

Abraham Cohen

The Johns Hopkins University
Baltimore, Maryland

PREFACE TO THE COMPLETELY REVISED EDITION

THE general plan of the original edition has been retained. However, important changes have been made in the order and in the manner of presentation of certain topics. Where experience has shown it to be desirable, exercises have been revised and their number increased. This is especially true of those affording interesting and important applications of both ordinary and partial differential equations to the Physical Sciences. The chapter dealing with solution by means of series has been largely rewritten.

Constructive suggestions for improving the original work have come to me from many sources. I am very grateful for these, and have taken advantage of many of them. I am especially indebted to Dr. John Williamson of the Johns Hopkins University and to Dr. Teresa Cohen of Pennsylvania State College, who have carefully read the proofs and have made most valuable suggestions. I am also deeply indebted to Mrs. Inez T. Loewus for material aid in the preparation of the manuscript.

A. C.

CONTENTS

CHAPTER PAGE

I. DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS..... 1

II. DIFFERENTIAL EQUATIONS OF THE FIRST ORDER...... 12

III. APPLICATIONS.............................. 46

IV. ADDITIONAL METHODS OF SOLUTION FOR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.................. 74

V. SINGULAR SOLUTIONS.......................... 86

VI. LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.................................. 101

VII. LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER 148

VIII. MISCELLANEOUS METHODS FOR SOLVING EQUATIONS OF HIGHER ORDER THAN THE FIRST................. 155

IX. INTEGRATION IN SERIES........................ 175

X. TOTAL DIFFERENTIAL EQUATIONS.................. 210

XI. SYSTEMS OF SIMULTANEOUS EQUATIONS............. 226

XII. PARTIAL DIFFERENTIAL EQUATIONS................ 242

XIII. PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER 250

XIV. PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER THAN THE FIRST............................ 275

NOTES

I. Condition that a Relation Exist between Two Functions of Two Variables............................ 302

II. Condition that n Functions of a Single Variable should Satisfy a Linear Identical Relation with Constant Coefficients.............................. 303

III. General Summary............................ 306

INDEX.. 309

ANSWERS...................................... 313

DIFFERENTIAL EQUATIONS

CHAPTER I

DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

I·1. Differential equation. A differential equation is an equation which involves differentials or one or more derivatives.

Equations of this type arise in the analytic statement of problems in many fields. For example, any geometric problem involving the slope or curvature of a curve, when expressed in analytic form, gives rise to a differential equation. Thus, if a curve is defined by the property that its tangent at any point (x, y) is perpendicular to the radius vector to the point, the analytic expression of this fact takes the form

$$(1) \qquad y' \equiv \frac{dy}{dx} = -\frac{x}{y}, \quad \text{or} \quad xdx + ydy = 0.$$

If the tangent is not perpendicular to, but makes a constant angle, α, with the radius vector, the analytic expression is

$$(2) \qquad \frac{y' - \dfrac{y}{x}}{1 + \dfrac{yy'}{x}} = m, \quad \text{where} \quad m = \tan\alpha.$$

Cleared of fractions, this may be written

$$(2') \quad mx + y - (x - my)y' = 0, \quad \text{or} \quad (mx + y)dx - (x - my)dy = 0.$$

If polar coördinates are employed, equation (2) takes the simpler form

$$(3) \qquad \frac{\rho}{\rho'} = m, \quad \text{or} \quad \frac{md\rho}{\rho} = d\theta.$$

The fact that all the tangents to a certain curve are at the same distance k from the origin is expressed by the differential equation

$$(4)\qquad \frac{y - xy'}{\sqrt{1 + y'^2}} = k, \quad \text{or} \quad (y - xy')^2 = k^2(1 + y'^2).$$

If a curve has the property that its radius of curvature is constant, the analytic expression of this fact takes the form

$$(5)\qquad \frac{(1 + y'^2)^{3/2}}{y''} = k, \quad \text{or} \quad k^2y''^2 = (1 + y'^2)^3.$$

In many fields, capable of being treated mathematically, differential equations arise when their laws are expressed in analytic form. Thus, consider the motion of a particle along a straight line, when subjected to a force which produces an acceleration proportional to the distance of the particle from a definite point on the line and always directed towards the point. The analytic expression of this law is

$$(6)\qquad \frac{d^2x}{dt^2} = -\mu^2x,$$

when the definite point is taken as the origin. Here μ is a constant.

In the field of chemistry we may cite as an example the fact that in the case of certain reactions the rate at which a substance is transformed is proportional to the quantity of the substance existent at that instant. If x is the amount of the substance that has been transformed up to a given time t, and a is the original amount of it, this law is expressed by the differential equation

$$(7)\qquad \frac{dx}{dt} = k(a - x),$$

where k is a constant.

The property that the normal at each point (x, y, z) of a surface should meet the plane $z = 0$ in a point $(x_0, y_0, 0)$ such that

$x_0/y_0 = x/y$ is expressed by

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0. \tag{8}$$

In the case of the motion of a stretched elastic string, the law of motion, under certain hypotheses, is given by the differential equation

$$\frac{\partial^2 y}{\partial t^2} - a^2\frac{\partial^2 y}{\partial x^2} = 0. \tag{9}$$

Here x is the coördinate measured along the line joining the extremities of the string, and y is the displacement of the point from its position of equilibrium.

I·2. Classification of differential equations. Differential equations, in which there is a single independent variable (and which, therefore, involve ordinary derivatives only), are known as *ordinary differential equations.* Examples of these are equations (1), (2), (3), (4), (5), (6), and (7).

If an equation involves more than one independent variable, so that partial derivatives enter, it is known as a *partial differential equation.* Equations (8) and (9) are of this type.

By the *order of a differential equation* we mean the order of the highest derivative involved. Thus, (1), (2), (3), (4), (7), and (8) are of the first order; (5), (6), and (9) are of the second order.

By the *degree of a differential equation* we mean the degree of the highest derivative entering, when the equation has been rationalized and cleared of fractions. Thus, (1), (2), (3), (6), (7), (8), and (9) are of the first degree; (4) and (5) are of the second degree.

I·3. Solution. By a *solution* of a differential equation we mean a relation,* connecting the independent and dependent variables, which satisfies the differential equation identically, i.e., for all values of the independent variable.

---

* A relation of this type is frequently called an *integral* of the differential equation, and the term *solution* is restricted to the form obtained by solving for the dependent variable, thus giving its value explicitly. This custom is quite general throughout Continental Europe. But the nomenclature employed in this book has been in consistent use in England and America for a long time.

The satisfying of the differential equation by a solution may be shown in various ways, as is illustrated below. The general method consists in calculating the values of the dependent variable and its derivatives, as functions of the independent variable, and substituting these in the differential equation. The result is an identity.

The relations $x^2 + y^2 = 1$ and $x^2 + y^2 = c$, where c is *any* constant, are solutions of (1), as may be verified at once by differentiating them.

Again, $\rho = e^{\theta/m}$ and $\rho = ce^{\theta/m}$, where c is *any* constant, are solutions of (3), as may be verified by differentiating and substituting the values of ρ and ρ' in the equation.

In the case of equation (4), $y - cx = k\sqrt{1 + c^2}$, where c is *any* constant, is readily seen to be a solution on differentiating and substituting the resulting value of y' in the equation. But $x^2 + y^2 = k^2$ is also a solution, as may be verified readily.

From the way in which the differential equation (5) was formed, it is evident that the equation of the family of circles $(x - a)^2 + (y - b)^2 = k^2$, where a and b are *any* constants, will satisfy (5). It will be left as an exercise for the student to verify this by differentiating twice, solving for y'' in terms of y' and y, and for y' in terms of x and y. Substituting these in the differential equation, one obtains either an identity or a relation consistent with the equation of the family of circles.

Differentiating twice, we see that $x = \sin \mu t$, or $x = \cos \mu t$, or $x = a \sin \mu t$, or $x = b \cos \mu t$, or $x = a \sin \mu t + b \cos \mu t$, where a and b are *any* constants, will satisfy equation (6).

In the case of the partial differential equation (8), $z = x^2 + y^2$ and $z = a\sqrt{x^2 + y^2} + b$, where a and b are *any* constants, and $z = f(x^2 + y^2)$, where $f(x^2 + y^2)$ is *any* function of $x^2 + y^2$, are readily seen to be solutions, by finding $\partial z/\partial x$ and $\partial z/\partial y$, and substituting them in the equation.

Similarly, $y = x + at$, $y = x - at$, $y = f(x + at) + \phi(x - at)$, where f and ϕ are *any* functions of their respective arguments $(x + at)$ and $(x - at)$, are solutions of (9).

From the illustrations given above we see that a differential equation has an indefinite number of solutions. This is exactly what may be expected, for the student has already learned to add a constant of integration when integrating a function. Thus, $\int \cos x dx = \sin x + c$, where c is the constant of integration.

The problem of integration studied in the Calculus is only a very special case of the general problem of solving a differential equation. To integrate $\int \cos x dx$ is to find a function, $\sin x + c$, whose derivative is $\cos x$. As a process in the Differential Equations, this amounts to finding $y = \sin x + c$ as the general solution of $y' = \cos x$.

A constant in a solution is said to be *arbitrary*, if any value whatever may be assigned to it. Thus, $y = \sin x + c$ is a solution of $y' = \cos x$, no matter what value may be assigned to c. In the illustrations of solutions given above, it should be noted that in the case of ordinary differential equations of the first order, the solutions mentioned involve either one or no arbitrary constant; for those of the second order, the solutions mentioned involve two, one, or no arbitrary constants.

To be distinguished from the arbitrary constants appearing in the solutions of differential equations are *literal* constants, such as m in (2) and (3), k in (4) and (5), μ in (6), k and a in (7), a in (9). While any desired value may be assigned* to each of these in stating the problem, once chosen, the value is fixed and that value alone can be used in the solution.

In the case of partial differential equations, arbitrary functions may appear in the solution, as illustrated above. The study of partial differential equations will be taken up in Chapter XII. For the present we shall restrict ourselves to the consideration of ordinary differential equations. The question naturally arises, what is the maximum number of arbitrary constants that may appear in the solution of a given ordinary differential equation?

---

* In specific problems, the values of these constants are prescribed. Thus, in (2) or (3) the value of m is known when that of α is given; in (6) the value of μ is determined by the nature of the apparatus used; and so on.

I·4. Derivation of the differential equation from its primitive. Starting with the relation

$$y = a \cos x, \tag{1}$$

where a is an arbitrary constant, we have, on differentiating it,

$$y' = -a \sin x. \tag{2}$$

Eliminating a between these two relations, we obtain

$$y' + y \tan x = 0, \tag{3}$$

which, by definition (§I·3), is a differential equation, having (1) for a solution.

But this is not the only differential equation that can have (1) for a solution. Thus, differentiating (2), we find

$$y'' = -a \cos x. \tag{4}$$

Eliminating a between (1) and (4), we obtain

$$y'' + y = 0, \tag{5}$$

which is a differential equation of the second order having (1) for a solution.

Or, eliminating a between (2) and (4), we find

$$y'' - y' \cot x = 0, \tag{6}$$

which is another differential equation having (1) for a solution.

Or, differentiating (3), we obtain still another differential equation of the second order

$$y'' + y' \tan x + y \sec^2 x = 0, \tag{7}$$

which must have (1) for a solution, since (3) does.

Again, differentiating (4) and eliminating a between the resulting relation and each one of (1), (2), or (4), we obtain three differential equations of the third order, each having (1) for a solution.

This process may be continued indefinitely.

Starting with

$$y = a \cos x + b \sin x, \tag{8}$$

where a and b are arbitrary constants, we have, on differentiating it,

$$y' = -a \sin x + b \cos x. \tag{9}$$

In general, in order to be able to eliminate two quantities, such as a and b, three relations involving them are required. In this case we must differentiate (9), obtaining a third relation

$$y'' = -a \cos x - b \sin x. \tag{10}$$

We can now eliminate a and b, and find the resulting differential equation

$$y'' + y = 0, \tag{11}$$

which, by definition, has (8) for solution.

Differentiating (10), we have

$$y''' = a \sin x - b \cos x. \tag{12}$$

Using this relation with (9), we find, on eliminating a and b,

$$y''' + y' = 0, \tag{13}$$

a differential equation of the third order having (8) for a solution.

This process also may be continued indefinitely.

In order to make the matter precise, we shall say that, starting with a given relation involving one or more arbitrary constants, *the* differential equation *corresponding to it as a primitive* is the *one of lowest order* which has it as a solution. Thus, (3) is *the* differential equation having (1) as a primitive, and (11) is *the* differential equation having (8) as a primitive.

In the general case, starting with a relation involving n independent* arbitrary constants, we obtain n additional relations by differentiating the relation n times. For, in general,† it is necessary and sufficient to have $(n + 1)$ independent relations from which to eliminate n independent quantities. This process is unambiguous. Owing to the fact that the primitive must be differentiated precisely n times, it results that *a primitive involving n independent arbitrary constants gives rise to a unique differential equation of the nth order.*

To illustrate, find the differential equations corresponding to the following primitives:

***Exercise* 1.** $y = ae^{-x} + be^{2x}$.

Here a and b are the arbitrary constants. Differentiating twice, we have

$$\begin{aligned} y' &= -ae^{-x} + 2be^{2x}, \\ y'' &= ae^{-x} + 4be^{2x}. \end{aligned}$$

From these three equations we must eliminate a and b. Adding the second and third, also the first and second, we obtain

$$y'' + y' = 6be^{2x}, \quad y' + y = 3be^{2x},$$

---

* The criterion as to when the arbitrary constants appearing in a relation are independent is too complicated to be given here. Frequently, inspection or other elementary means will enable one to detect when a set of constants is not independent. For example, in $y = ae^b \sin x$, ae^b is no more general than the single arbitrary constant a. Again, in the relation $ax^2 - 2by + c = 0$, only two independent constants appear, and not three, for an arbitrary choice of all three is no more general than an arbitrary choice of the ratios of two of them to the third. Again, the relation $ax^2 - 2bxy + y^2 = 0$, when solved for y, viz.

$$y = (b \pm \sqrt{b^2 - a})x,$$

is seen to involve only one arbitrary constant, not two, for the expression is no more general than

$$y = cx.$$

† A precise statement of the conditions under which this is true is too complicated to be stated here.

two relations involving b only. Eliminating b between these, we have

$$y'' + y' = 2(y' + y), \quad \text{or} \quad y'' - y' - 2y = 0,$$

the desired differential equation.

Exercise **2.** $(x - c)^2 + y^2 = r^2$.

Here c is the arbitrary constant. Differentiating once, we have

$$x - c + yy' = 0.$$

From these two equations we must eliminate c. Now

$$x - c = -yy'.$$

Substituting this in the original equation, we have

$$y^2y'^2 + y^2 = r^2.$$

3. $y = cx + \sqrt{1 - c^2}$.

4. $(x - a)^2 + (y - b)^2 = r^2$ [a and b are the arbitrary constants].

5. $y = c_1x^2 + c_2$.

6. $x^2 + y^2 + cx = 0$.

7. $c_1x^2 + c_2y^2 = 1$.

8. $y = ae^{kx} + be^{lx}$ [a and b are the arbitrary constants].

9. $x^2 = 2cy + c^2$.

10. $y^2 = 4c(x + c)$.

11. $y = \sin(x + c)$.

12. $y = \sin cx$.

I·5. General and particular solutions. If we start with a differential equation, its solution involving the maximum number of independent arbitrary constants is the primitive which gives rise to the differential equation. For, this solution cannot contain more than n arbitrary constants, by the theorem in §I·4. Besides, it must contain as many as n; otherwise it would be the primitive of a lower ordered equation.

A solution involving the maximum number of arbitrary constants is called the *general* (or *complete*) *solution.*

A solution which is derivable from the general solution by assigning fixed values to the arbitrary constants is called a *particular solution.* Thus, $y = \cos x$ and $y = \cos x - \sin x$ are particular solutions of $y'' + y = 0$.

As mentioned in §I·3, the problem of the Differential Equations includes that of the Integral Calculus as a special case. Thus, in the latter, the general problem consists in solving

$$\frac{dy}{dx} = f(x).$$

This is only a special case of the problem of finding the solution of the differential equation of the first order involving two variables,

$$\frac{dy}{dx} = f(x, y),$$

where $f(x, y)$ may be a function of both the variables. We speak of *integrating* or *solving the equation* in the general case, and at times refer to the simpler problem of the Integral Calculus as that of performing a *quadrature.*

While the problem of finding the differential equation corresponding to a given primitive is a direct one, and can be carried out according to a general plan, involving only differentiation and elimination, there is no practical general method for the inverse problem of finding the primitive or general solution of a given differential equation.

In the following chapters we shall discuss, in as systematic a manner as possible, some of the classes of equations whose general solutions can be found. In this elementary treatise we shall be concerned primarily with those types of differential equations whose solutions can be obtained by means of quadratures.

In actual practice, differential equations arise which do not yield to these elementary methods. Often the powerful methods of the theory of functions are required to treat such cases. These are beyond the scope of this treatise. However, we shall consider some of the simpler types of differential equations whose solutions can be obtained as infinite series.

In many cases, where the differential equation arises in studying a special problem, it is some particular solution satisfying the peculiar conditions of the special problem that is desired.

When the general solution can be found, it is often possible to obtain from it the desired particular solution. However, such special solutions can be found at times, even when it is not possible or practicable to find the general solution.

Again, especially in the case of partial differential equations where the general solutions involve arbitrary functions, the latter are so general in character, that the desired particular solutions cannot always be obtained readily from the general solutions. In such cases, the desired special solutions are sought independently of the general solutions.

CHAPTER II

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

II·1. Variables separated or separable. A differential equation of the first order, when solved for dy/dx, takes the form

$$\frac{dy}{dx} = F(x, y),$$

or, when cleared of fractions, it may be written in the more symmetrical form

$$M(x, y)dx + N(x, y)dy = 0.$$

If M* is a function of x only, and N one of y only, i.e., if

$$M(x)dx + N(y)dy = 0,$$

the variables are said to be *separated.* In this case the solution is obtained by integrating each term and equating the sum of the integrals to an arbitrary constant, thus

$$\int M(x)dx + \int N(y)dy = c.$$

Frequently the variables can be separated readily. This is the case when $M = f_1(x)f_2(y)$ and $N = \phi_1(x)\phi_2(y)$. For, divided by $\phi_1(x)f_2(y)$, the equation takes the form

$$\frac{f_1(x)}{\phi_1(x)}\,dx + \frac{\phi_2(y)}{f_2(y)}\,dy = 0,$$

in which the variables are separated.

---

* Throughout this book it will be understood that a function of a specified set of variables *need not* involve all of them. *It may be a constant,* not involving any of them. Thus $M(x, y)$ may involve both x and y, it may involve only one of them, or it may be a constant and not involve either. Similarly, $M(x)$ may involve x or it may be a constant; but it does not involve y.

The solution may frequently be expressed in one of several forms. No general rule can be given as to which is the most desirable. In general, however, we may say that an algebraic form is preferable to one involving transcendental functions, and that a rational form is preferable to one involving radicals. Moreover, it is preferable to replace a function of the arbitrary constant by a single letter when possible.

Exercise **1.** $xydx - (1 + x^2)dy = 0.$

Dividing by $y(1 + x^2)$, we obtain

$$\frac{x}{1 + x^2}dx - \frac{dy}{y} = 0.$$

Integrating, we have

$$\tfrac{1}{2}\log(1 + x^2) - \log y = c,$$

or

(1) $$\log\frac{\sqrt{1 + x^2}}{y} = c,$$

which is the general solution.

Since, by the definition of the Naperian or natural logarithm, which we are using here, $e^{\log u} = u$, we can write (1) in the form

(2) $$\frac{\sqrt{1 + x^2}}{y} = e^c.$$

Here our variables enter algebraically. Since c is an arbitrary constant, e^c is also one, and we may represent it by a single letter C. The solution can then be put in the form

(3) $$\sqrt{1 + x^2} = Cy.$$

Or we may replace e^c by the reciprocal of an arbitrary constant and write it $1/k$. Then the general solution takes the form

(4) $$y = k\sqrt{1 + x^2}.$$

Moreover, replacing C^2 by c_1, and k^2 by c_2, we can write (3) and (4) in the forms

(5) $$1 + x^2 = c_1y^2 \quad\text{and}\quad y^2 = c_2(1 + x^2)$$

respectively.

Any of these forms, (1), (2), (3), (4), or (5), may be selected as the general solution.

***Exercise* 2.** $(1 + y^2)dx + (1 + x^2)dy = 0.$

Separating the variables, we have

$$\frac{dx}{1 + x^2} + \frac{dy}{1 + y^2} = 0.$$

Integrating, we obtain

$$\tan^{-1} x + \tan^{-1} y = c.$$

We can express this in terms of algebraic functions of x and y by taking the tangent of each member, thus

$$\frac{x + y}{1 - xy} = \tan c, \quad \text{or simply,} \quad \frac{x + y}{1 - xy} = c_1.$$

The student should show that another form for the general solution is

$$y = \frac{c_1 - x}{1 + c_1 x}.$$

***Exercise* 3.** $e^x y^3 dx - x^2 dy = 0.$

Separating the variables and integrating, we find the solution in the form

$$\int \frac{e^x}{x^2}\, dx + \frac{1}{2y^2} = c.$$

The result of the first quadrature involved here cannot be expressed in terms of elementary functions. At times, in a case of this sort, when an approximate solution is sought satisfying, to within a given error, the conditions of a given problem, it is convenient to express the result of the quadrature as an infinite series. Here, noting that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

we can write the solution in the form*

$$-\frac{1}{x} + \log x + \frac{x}{2!} + \frac{x^2}{2 \cdot 3!} + \frac{x^3}{3 \cdot 4!} + \cdots + \frac{1}{2y^2} = c.$$

---

* Integration, term by term, is permissible in the case of a series whose terms are integral powers of the variable.

4. $xy\,dx + \sqrt{1+x^2}\,dy = 0.$

5. $(x - xy^2)dx + (y + x^2y)dy = 0.$

6. $(y + 3)dx + \cot x\,dy = 0.$

7. $\sin x \cos y\,dx + \cos x \sin y\,dy = 0.$

8. $\sin x \cos^2 y\,dx + \cos^2 x\,dy = 0.$

II·2. Homogeneous equation. A frequently occurring class of equations in which the variables can be separated, not by inspection, but after a simple transformation of variables, is one in which M and N are homogeneous functions of x and y and of the same degree. Then M/N is a homogeneous function of degree zero, and is, therefore, a function of y/x.*

Our equation can now be written

$$\frac{dy}{dx} = -\frac{M}{N} \doteq F\left(\frac{y}{x}\right). \tag{1}$$

Let y/x be a new variable, viz. v. Then

$$y = vx, \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx},$$

and the differential equation takes the form

$$v - F(v) + x\frac{dv}{dx} = 0, \tag{2}$$

in which the variables are separable.

---

* A *homogeneous function* of x and y of degree r is such a function of these variables that, when in it x and y are replaced by tx and ty respectively, where t is any multiplier whatever, the result will be the original function multiplied by t^r. (This definition can be extended at once to a function of any number of variables. It is obviously consistent with the definition of a homogeneous polynomial.) This definition can be formulated thus: If $f(x, y)$ is a homogeneous function of degree r, then

$$f(tx, ty) \equiv t^r f(x, y),$$

where the symbol $\equiv$ is used to indicate that the relation is not merely an equation, but an identity, holding for all values of x and y.

If, now, we put $t = 1/x$, we have

$$f\left(1, \frac{y}{x}\right) \equiv \frac{1}{x^r} f(x, y), \quad \text{or} \quad f(x, y) \equiv x^r f\left(1, \frac{y}{x}\right) \equiv x^r F\left(\frac{y}{x}\right).$$

In particular when $r = 0$, we have $f(x, y) \equiv F\left(\frac{y}{x}\right)$.

Compare §113 of the author's Calculus.

Integrating this, and then replacing v by its value in terms of x and y, we have the general solution.

***Exercise* 1.** $(xe^{y/x} + y)dx - xdy = 0.$

Put $y = vx$. Then $dy = vdx + xdv$, and the differential equation takes the form

$$x(e^v + v)dx - xvdx - x^2dv = 0,$$

or

$$\frac{dx}{x} - \frac{dv}{e^v} = 0.$$

Integrating, we have

$$\log x + e^{-v} = c, \quad \text{or} \quad \log x + e^{-y/x} = c.$$

***Exercise* 2.** $x\frac{dy}{dx} + y = 2\sqrt{xy}.$

Put $y = vx$. Then $dy/dx = v + x\,dv/dx$, and the equation takes the form

$$x^2\frac{dv}{dx} + 2xv = 2x\sqrt{v},$$

or

$$\frac{dv}{2(v - \sqrt{v})} + \frac{dx}{x} = 0.$$

To integrate the first term, let $\sqrt{v}$ be a new variable. Performing the integration, we have

$$\log(\sqrt{v} - 1) + \log x = c. \tag{3}$$

A simple reduction leads to

$$x\sqrt{v} - x = e^c, \quad \text{or} \quad \sqrt{xy} - x = c_1.$$

Rationalizing, we have finally

$$x^2 - xy + 2cx + c^2 = 0. \tag{4}$$

Here c is used instead of c_1 for the arbitrary constant. It is not the same constant c that appears in (3). Since (4) is an alternative form for the solution, there is no danger of confusion, and certainly no error, in using c, the initial letter in the word constant, to represent the arbitrary constant in either of the expressions for the general solution.

3. $(y^2 - xy)dx + x^2dy = 0.$ **4.** $x\frac{dy}{dx} - y + \sqrt{y^2 - x^2} = 0.$

5. $(2x^2y + 3y^3)dx - (x^3 + 2xy^2)dy = 0.$

6. $\left(x + y\cos\frac{y}{x}\right)dx - x\cos\frac{y}{x}\,dy = 0.$

7. $(x^2 + y^2)dx - axydy = 0.$

8. $(y^2 - x^2 + 2mxy)dx + (my^2 - mx^2 - 2xy)dy = 0.$

II·3. Equation in which M and N are linear but not homogeneous. If M and N are both of the first degree, but not homogeneous, we can, in general, make them homogeneous by a transformation of variables. If the equation has the form

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0,$$

put

$$x = X + \alpha, \quad y = Y + \beta.$$

The equation then takes the form

$$(a_1X + b_1Y + a_1\alpha + b_1\beta + c_1)dX \\ + (a_2X + b_2Y + a_2\alpha + b_2\beta + c_2)dY = 0.$$

If α and β are so chosen that

$$a_1\alpha + b_1\beta + c_1 = 0 \quad \text{and} \quad a_2\alpha + b_2\beta + c_2 = 0,$$

the equation becomes homogeneous, and the method of §II·2 applies.

Exercise 1. $(4x + 3y + 1)dx + (x + y + 1)dy = 0.$

The equations

$$4\alpha + 3\beta + 1 = 0 \quad \text{and} \quad \alpha + \beta + 1 = 0$$

have the solution

$$\alpha = 2 \quad \text{and} \quad \beta = -3.$$

Hence the transformation

$$x = X + 2, \quad y = Y - 3$$

reduces the differential equation to the homogeneous form

$$(4X + 3Y)dX + (X + Y)dY = 0.$$

Putting $Y = vX$, we obtain

$$\frac{1 + v}{(2 + v)^2}\,dv + \frac{dX}{X} = 0.$$

Noting that

$$\frac{1+v}{(2+v)^2}=\frac{2+v-1}{(2+v)^2}=\frac{1}{2+v}-\frac{1}{(2+v)^2},$$

we can easily perform the quadratures involved. Doing this, we have

$$\log(2+v)+\frac{1}{2+v}+\log X=c,$$

or

$$\log X(2+v)=c-\frac{1}{2+v};$$

whence

$$\log(2X+Y)=c-\frac{X}{2X+Y}.$$

Passing back to x and y, we find

$$\log(2x+y-1)=c-\frac{x-2}{2x+y-1}.$$

2. $(4x-y+2)dx+(x+y+3)dy=0.$

3. $(2x-y)dx+(x-2y-3)dy=0.$

This method breaks down when $a_1b_2-a_2b_1=0$. But, in this case, we can find a transformation which will separate the variables at once. For we have $a_2/a_1=b_2/b_1=k$, a constant, and the equation takes the form

$$(a_1x+b_1y+c_1)dx+[k(a_1x+b_1y)+c_2]dy=0.$$

If, now, we put $a_1x+b_1y=v$, whence $dy=(dv-a_1dx)/b_1$, our equation takes the form

$$b_1(v+c_1)dx+(kv+c_2)(dv-a_1dx)=0,$$

from which the variable x is absent, and consequently the variables are separable; for it is evident that, *whenever one of the variables is absent from both M and N, the variables are separable.*

4. $(2x+y)dx-(4x+2y-1)dy=0.$

5. $(2x-y+3)dx+(4x-2y+7)dy=0.$

6. $(15x+6y-7)dx+(5x+2y-3)dy=0.$

★**II·4. Isobaric equation.** Extending the definition of homogeneous function given in the footnote of §II·2, we shall define $f(x, y, y')$* as an *isobaric function of weight* r, if a number m can be found such that the result of replacing x by tx, y by $t^m y$, y' by $t^{m-1}y'$ is the original function multiplied by t^r; that is, if

$$f(tx, t^m y, t^{m-1}y') \equiv t^r f(x, y, y').$$

In particular, if $t = 1/x$, this reduces to

$$f\left(1, \frac{y}{x^m}, \frac{y'}{x^{m-1}}\right) \equiv \frac{1}{x^r} f(x, y, y'),$$

or

(1) $$f(x, y, y') \equiv x^r f\left(1, \frac{y}{x^m}, \frac{y'}{x^{m-1}}\right),$$

or, finally

(2) $$f(x, y, y') \equiv x^r F\left(\frac{y}{x^m}, \frac{y'}{x^{m-1}}\right);$$

i.e., the isobaric function may be written as the product of x^r and a function of y/x^m and y'/x^{m-1}.

A differential equation obtained by equating an isobaric function of the form (1) to zero will be said to be an *isobaric differential equation.* Given such an equation, we see from (2) that it can be put into the form

$$F\left(\frac{y}{x^m}, \frac{y'}{x^{m-1}}\right) = 0.$$

Solving this for y'/x^{m-1}, and then multiplying by x^{m-1}, we have the equation in the form

(3) $$y' \equiv \frac{dy}{dx} = x^{m-1}\phi\left(\frac{y}{x^m}\right).$$

★ Sections indicated by a ★ may be omitted in a short course.

* Here, as throughout this book, y' denotes $\frac{dy}{dx}$.

Introducing the new variable $v = y/x^m$, *whence* $y = vx^m$, *we obtain the equation in the form*

$$x^m \frac{dv}{dx} + mx^{m-1}v = x^{m-1}\phi(v),$$

in which the variables are separable.

The homogeneous differential equation, considered in §II·2, is a special case of this type of equation, when $m = 1$.

A little experience will enable one to detect when a given expression in x, y, and y' is isobaric. If the weights m and $m - 1$ are assigned to y and y' respectively, the term $x^a y^b y'^c$ evidently has the weight $a + bm + c(m - 1)$, where a, b, and c are any numbers, positive or negative, integral or fractional. The weight of x is always 1.

If an expression is to be isobaric, all its terms must be isobaric and of the same weight. Thus the terms of

$$2x^2y' - x^2y^2 + 2xy + 1$$

have the weights

$$2 + (m - 1) \equiv m + 1, \quad 2 + 2m, \quad 1 + m, \quad 0$$

respectively. Each of these is equal to zero if $m = -1$. Hence, for $m = -1$, the expression is isobaric of weight zero.

Again, the terms of the differential equation

$$x^4y'^2 - xy' - y = 0 \tag{4}$$

have the weights

$$4 + 2(m - 1) \equiv 2m + 2, \quad 1 + (m - 1) \equiv m, m$$

respectively. Setting these equal to each other, we find that each of them equals -2 if $m = -2$. Hence, for $m = -2$, the equation (4) is seen to be isobaric of weight -2.

If the equation is solved for $2y'$, it becomes

$$2y' = \frac{1}{x^3} + \frac{\sqrt{1 + 4x^2y}}{x^3}, \tag{5}$$

and is still isobaric for $m = -2$, as may be seen by replacing x by tx, y by y/t^2, y' by y'/t^3. Or we may note that x^2y has the weight 0; hence, $1 + 4x^2y$ has the weight 0, and

$\sqrt{1 + 4x^2y}/x^3$ has the weight -3, which is also the weight of y' and $1/x^3$.

Since the differential equation is isobaric for $m = -2$, the introduction of the new variable $v = x^2y$ enables one to separate the variables.

In the case of the differential equation

$$(6) \qquad 2xyy' = y^2 + \sqrt{y^4 - 4x^2}$$

the term $\sqrt{y^4 - 4x^2}$ is isobaric if $4m$, the weight of y^4, is equal to 2, the weight of x^2; i.e., if $4m = 2$, or $m = \frac{1}{2}$, in which case the term has the weight 1. For $m = \frac{1}{2}$, each of the other terms also has the weight 1. Hence the introduction of the new variable $v = y/\sqrt{x}$ enables one to separate the variables. This being true, the introduction of the square of this as a new variable, namely, $v = y^2/x$, will be equally effective and much simpler in execution. See Exercise 6 below.

***Exercise* 1.** $2x^2y' - x^2y^2 + 2xy + 1 = 0.$

Since this was found to be isobaric when $m = -1$, let $v = xy$. Then $y = v/x$ and $x^2y' = xv' - v$, and the differential equation becomes

$$2x\frac{dv}{dx} - v^2 + 1 = 0.$$

Separating the variables, we have

$$\frac{2\,dv}{1 - v^2} + \frac{dx}{x} = 0.$$

Integrating this, we obtain

$$\log\left(\frac{1 + v}{1 - v}\right) + \log x = c, \quad \text{or} \quad \frac{x(1 + v)}{1 - v} = c_1.$$

Replacing v by its value, we have the solution in the form

$$x + x^2y = c_1(1 - xy).$$

2. $x^3y' - x^2y + y^2 = 0.$ **3.** $x^3y' + 4x^2y + 1 = 0.$

4. $(x + 2x^2y)y' + 2y + 3xy^2 = 0.$

5. $2x^3y' = 1 + \sqrt{1 + 4x^2y}.$ **6.** $2xyy' = y^2 + \sqrt{y^4 - 4x^2}.$

7. Show that $yf_1(xy) + xf_2(xy)y'$ is isobaric for $m = -1$. Here $f_1(xy)$ and $f_2(xy)$ are functions of the product xy.

II·5. Equation of the form $y' = F(ax + by + c)$. Another type of equation, in which the variables can be separated by the introduction of a new dependent variable, is of the form

$$y' = F(ax + by + c), \tag{1}$$

which is characterized by the fact that y' is equal to a function of the *linear* expression $ax + by + c$. Put $ax + by + c = v$. Then $by' = v' - a$, and the differential equation becomes

$$v' \equiv \frac{dv}{dx} = a + bF(v), \tag{2}$$

in which the variables are separable.

1. $(x + y)^2dx - dy = 0$. **2.** $y' = (x - y + a)^2$.
3. $(2x + y - 2)y' = 4$. **4.** $y' = \sin^2(x - y)$.

II·6. Exact differential equation. Integrating factor. Up to this point we have employed the very serviceable method of solving a differential equation of the first order, by which the introduction of a new dependent variable transforms the equation into one in which the variables are separable. However, this method is not always practicable. A second and also widely used method results from the following considerations:

If the general solution of the differential equation

$$Mdx + Ndy = 0 \tag{1}$$

is solved for the constant of integration, it takes the form

$$u(x, y) = c. \tag{2}$$

The differential equation having this as primitive is evidently

$$du \equiv \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0.^* \tag{3}$$

---

* In the case of a function of a single variable, $y = f(x)$, the differential of y is defined as

$$dy \equiv f'(x)dx.$$

So, in the case of a function of several variables, $u = f(x, y)$, the differential of u is defined as

$$du \equiv \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

For example, see the author's Calculus, §106.

Only one differential equation of the first order can correspond to a primitive involving a single arbitrary constant. Hence, either (1) and (3) must be identical, or their coefficients must be proportional, i.e.,

(4)
$$\frac{\dfrac{\partial u}{\partial x}}{M} = \frac{\dfrac{\partial u}{\partial y}}{N} = \mu(x, y).$$

This common ratio is, at most, a function of x and y; it may be a function of one of the variables only, or simply a constant. Calling it μ, we have

$$\frac{\partial u}{\partial x} = \mu M, \quad \frac{\partial u}{\partial y} = \mu N;$$

whence

(5)
$$\mu(Mdx + Ndy) \equiv du.$$

We shall speak of an expression which is the differential of a function of one or more variables as an *exact differential.* Thus, $\mu(Mdx + Ndy)$ in (5) is exact, since it is the differential of a function $u(x, y)$. We shall further speak of a differential equation as an *exact differential equation* if, when all the terms in it are brought to one side, that member is an exact differential.* The above result can now be stated as follows: *Whenever the general solution of the differential equation* (1) *can be written in the form* (2), *a factor* $\mu(x, y)$ *exists which, when introduced, will make the equation exact.*

This multiplier is known as an *integrating factor* because, as

---

* A differential equation in the form

$$M + N\frac{dy}{dx} = 0 \quad \text{or} \quad M + Ny' = 0$$

will also be said to be exact, if the left-hand member is the total derivative of some function $u(x, y)$ with respect to x. Thus, since $y + xy'$ is the derivative of xy with respect to x, the differential equation

$$y + xy' = 0$$

is said to be exact as well as the equation

$$ydx + xdy = 0.$$

we shall see (§II·8), when the equation is exact, its integration can be effected by a general method.

It is interesting to note that *once an integrating factor has been found for the differential equation* (1), *an indefinite number of them can be found* (whenever the quadratures involved in determining u can be effected). Suppose that μ is an integrating factor. Then

$$\mu(Mdx + Ndy) \equiv du(x, y),$$

where the sign of identity $\equiv$ means that $\mu M = \partial u/\partial x$ and $\mu N = \partial u/\partial y$.

Now, if $F(u)$ is any function of u, whose indefinite integral, $\phi(u)$, is known, we have

$$\mu F(u) Mdx + \mu F(u) Ndy = F(u)\frac{\partial u}{\partial x}dx + F(u)\frac{\partial u}{\partial y}dy = F(u)du = d\phi(u).$$

Hence, $\mu F(u)$ is also an integrating factor. Since $F(u)$ may be chosen in an indefinite number of ways, we see that the number of integrating factors is infinite. Thus, it is obvious on inspection that $xdy - ydx = 0$ has $1/x^2$ for an integrating factor. We have, actually, $(xdy - ydx)/x^2 = d(y/x)$. Here $\mu = 1/x^2$, $u = y/x$. Then $(1/x^2)F(y/x)$ is an integrating factor. In particular, more or less obvious integrating factors are obtained by putting

$$F\left(\frac{y}{x}\right) = \frac{x}{y}, \quad \text{or} \quad \frac{x^2}{y^2}, \quad \text{or} \quad \frac{x^2}{x^2 + y^2}, \quad \text{or} \quad \frac{2x^2}{x^2 - y^2},$$

giving, respectively, the exact differentials

$$d \log\left(\frac{y}{x}\right), \quad d\left(-\frac{x}{y}\right), \quad d \tan^{-1}\frac{y}{x}, \quad d \log\frac{x + y}{x - y}.$$

II·7. Condition that a differential equation be exact. If the equation

$$Mdx + Ndy = 0 \tag{1}$$

is exact, a function u exists such that

$$Mdx + Ndy \equiv du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy,$$

or

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}. \tag{2}$$

From these relations it follows that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{3}$$

For, from (2), we have on differentiating

$$\text{(4)} \qquad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

But, in general, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$. Hence, the exactness of equation (1) leads to the relation (3), which is thus seen to be a *necessary condition* for the exactness of equation (1).

It is also a *sufficient condition* for the exactness of (1). In proving this, we shall actually find a function u whose differential is $Mdx + Ndy$ when the relation (3) holds. That is, assuming (3), we shall find a function u such that

$$\text{(2)} \qquad \frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial u}{\partial y} = N.$$

In order that the first equation of (2) may be satisfied, we must have

$$\text{(5)} \qquad u = \int^{(x)} M dx + Y(y),^*$$

where Y is a function of y only, and plays the rôle of a constant of integration, since y is considered a constant in the process of integration involved in (5). The value of u given by (5) will satisfy the first equation of (2), provided $Y(y)$ is not a function of x. It may be any function of y. In order that u satisfy the second equation of (2), we must have

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \int^{(x)} M dx + \frac{dY}{dy} = N;$$

that is, Y must satisfy the equation

$$\text{(6)} \qquad \frac{dY}{dy} = N - \frac{\partial}{\partial y} \int^{(x)} M dx.$$

Since the left-hand member of (6) is a function of y only, the same must be true of the right-hand member; that is, the latter must be free of x, or, in other words, it must be a constant as far as x is concerned, and its derivative with respect to x must be zero. As a matter of fact, that derivative is $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$,* which is zero because of (3). We can then find Y to satisfy (6) by a quadrature, viz.,

---

* By $\int^{(x)} Mdx$ we mean the result of integrating Mdx considering y as a constant.

$$Y = \int \left[N - \frac{\partial}{\partial y} \int^{(x)} M\,dx \right] dy,$$

and the resulting value of u in (5) will satisfy both equations (2).

Exercise. Using the fact that (3) is the necessary and sufficient condition for exactness, prove that if μ is an integrating factor, such that $\mu(M\,dx + N\,dy) \equiv du$, $\mu F(u)$ is also an integrating factor. Compare §II·6, small type, also Theorem II below.

Several other facts may be mentioned here:

Theorem I. *If $u(x, y) =$ const. is a solution of the equation*

$$M\,dx + N\,dy = 0, \tag{1}$$

then u satisfies the equation

$$N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y} = 0; \tag{7}$$

and conversely.

For, as noted in §II·6, if $u =$ const. is a solution of (1), the differential equation

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0 \tag{8}$$

must be the same as equation (1), their left-hand members differing possibly by a factor which does not involve dy/dx; that is,

$$\frac{\dfrac{\partial u}{\partial x}}{M} = \frac{\dfrac{\partial u}{\partial y}}{N}, \tag{9}$$

---

* Since, in general, the order of differentiation is immaterial,

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\int^{(x)} M\,dx = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\int^{(x)} M\,dx.$$

Moreover, $\frac{\partial}{\partial x}\int^{(x)} M\,dx = M$, since, in both of the processes involved, y is considered constant.

or

$$N\frac{\partial u}{\partial x} = M\frac{\partial u}{\partial y}. \tag{7}$$

To prove the converse, we note that if u satisfies (7), the relation (9) follows. But (9) means that the equations (1) and (8) have the same solutions, and $u =$ const. is evidently a solution of (8).

We can now prove

Theorem II. *If μ_1 and μ_2 are two independent* integrating factors of the equation* (1), *$\mu_1/\mu_2 =$ const. is a solution of* (1).

Since μ_1 and μ_2 are integrating factors of (1), we have

$$\frac{\partial(\mu_1 M)}{\partial y} - \frac{\partial(\mu_1 N)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial(\mu_2 M)}{\partial y} - \frac{\partial(\mu_2 N)}{\partial x} = 0,$$

or

$$M\frac{\partial \mu_1}{\partial y} - N\frac{\partial \mu_1}{\partial x} + \mu_1\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 0$$

and

$$M\frac{\partial \mu_2}{\partial y} - N\frac{\partial \mu_2}{\partial x} + \mu_2\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 0.$$

Multiplying these by μ_2 and μ_1 respectively, and taking the difference, we have

$$M\left(\mu_2\frac{\partial \mu_1}{\partial y} - \mu_1\frac{\partial \mu_2}{\partial y}\right) - N\left(\mu_2\frac{\partial \mu_1}{\partial x} - \mu_1\frac{\partial \mu_2}{\partial x}\right) = 0.$$

Dividing by μ_2^2, we obtain

$$M\frac{\partial}{\partial y}\left(\frac{\mu_1}{\mu_2}\right) - N\frac{\partial}{\partial x}\left(\frac{\mu_1}{\mu_2}\right) = 0.$$

Comparing this with equation (7) and using the previous theorem, we see that $\mu_1/\mu_2 =$ const. is a solution of equation (1).

II·8. Exact differential equation. The proof of the sufficiency of the condition (3) of §II·7 suggests a method of solving an exact differential equation. For, in this case $Mdx + Ndy \equiv du$, where

$$u = \int^{(x)} Mdx + \int\left[N - \frac{\partial}{\partial y}\int^{(x)} Mdx\right] dy.$$

Hence, the general solution is

$$\int^{(x)} Mdx + \int\left[N - \frac{\partial}{\partial y}\int^{(x)} Mdx\right] dy = c.$$

---

* Independence implies that μ_2 is not equal to μ_1 multiplied by a constant.

This formula appears quite complicated; but, in actual practice, the processes involved are not difficult to remember.

1° First we evaluate $\int^{(x)} M dx$.

2° Then we take the partial derivative, with respect to y, of this partial integral, with respect to x, and subtract it from N, thus

$$N - \frac{\partial}{\partial y}\int^{(x)} M dx \equiv R(y).$$

3° This remainder, $R(y)$, is either zero, a constant, or at most a function of y. Hence the solution of the differential equation is

$$\int^{(x)} M dx + \int R(y) dy = c.$$

As an exercise the student should show that the general solution may also be obtained in the form

$$\int^{(y)} N dy + \int \left[M - \frac{\partial}{\partial x}\int^{(y)} N dy \right] dx = c.$$

***Exercise* 1.** $\frac{2xy + 1}{y} dx + \frac{y - x}{y^2} dy = 0.$

Since $\frac{\partial}{\partial y}\left(\frac{2xy+1}{y}\right) = -\frac{1}{y^2} = \frac{\partial}{\partial x}\left(\frac{y-x}{y^2}\right)$, the equation is exact. Here

$$\int^{(x)} \frac{2xy + 1}{y} dx = x^2 + \frac{x}{y}.$$

From this we find readily $R(y) = \frac{1}{y}$, and $\int R(y) dy = \log y$. Hence

$$x^2 + \frac{x}{y} + \log y = c$$

is the general solution.

***Exercise* 2.** $\frac{y^2 - 2x^2}{xy^2 - x^3} dx + \frac{2y^2 - x^2}{y^3 - x^2 y} dy = 0.$

It will be left as an exercise to show that this is exact. Here

$$\int^{(x)} \frac{y^2 - 2x^2}{xy^2 - x^3}\,dx = \int^{(x)} \frac{(y^2 - x^2) - x^2}{x(y^2 - x^2)}\,dx = \int^{(x)} \frac{dx}{x} - \int^{(x)} \frac{x\,dx}{y^2 - x^2}$$
$$= \log x + \tfrac{1}{2}\log(y^2 - x^2).$$

The partial derivative of this with respect to y is $y/(y^2 - x^2)$. Hence

$$R(y) = N - \frac{y}{y^2 - x^2} = \frac{1}{y},$$

and the solution is

$$\log x + \tfrac{1}{2}\log(y^2 - x^2) + \log y = c_1, \quad \text{or} \quad x^2y^2(y^2 - x^2) = c.$$

***Exercise* 3.** $\dfrac{dx}{\sqrt{x^2 + y^2}} + \left(\dfrac{1}{y} - \dfrac{x}{y\sqrt{x^2 + y^2}}\right) dy = 0.$

This should be shown to be exact. Here

$$\int^{(x)} \frac{dx}{\sqrt{x^2 + y^2}} = \log(x + \sqrt{x^2 + y^2}).$$

The partial derivative of this with respect to y is

$$\frac{y}{\sqrt{x^2 + y^2}\,(x + \sqrt{x^2 + y^2})}.$$

Multiplying numerator and denominator of this by $x - \sqrt{x^2 + y^2}$, we find that it reduces to the value of N. Hence, $R(y) = 0$, and the solution is

$$\log(x + \sqrt{x^2 + y^2}) = c_1, \quad \text{or} \quad y^2 + 2cx = c^2.$$

4. $(x + y)dx + x\,dy = 0.$

5. $(6x - 2y + 1)dx - (2x - 2y + 3)dy = 0.$

6. $\dfrac{(x + y)dx - (x - y)dy}{x^2 + y^2} = 0.$ **7.** $\dfrac{dx}{\sqrt{xy}} + \left(\dfrac{2}{y} - \sqrt{\dfrac{x}{y^3}}\right) dy = 0.$

8. $x^{\frac{1}{2}} y^{\frac{1}{2}} [(5xy - 3y^3)dx + (3x^2 - 7xy^2)dy] = 0.$

9. $(2xy^{-3} - 3x^2)dx + 3(1 - x^2)y^{-4}dy = 0.$

10. $\left(2x \sin\dfrac{y}{x} - y \cos\dfrac{y}{x}\right) dx + x \cos\dfrac{y}{x}\,dy = 0.$

II·9. Linear differential equation of the first order. A linear differential equation (of any order) is one which is of the first degree in the dependent variable and all of its derivatives. Thus, the general type of a *linear differential equation of the first order* is

$$X_0y' + X_1y = X_2,$$

where the coefficients X_0, X_1, X_2 are functions of x only. This includes the possibility of any of them being constant. Divided by the coefficient of y', this may be written in the form

$$y' + Py = Q \quad \text{or} \quad \frac{dy}{dx} + Py = Q, \tag{1}$$

where P and Q are functions of x only.

Assuming the existence of an integrating factor for this differential equation, one can show readily that $e^{\int P dx}$ is such a factor. In doing this, we shall endeavor to find an integrating factor $\mu(x)$, which is a function of x only. Our problem, then, is to find $\mu(x)$ such that

$$\mu y' + \mu P y - \mu Q = 0$$

will be exact.

In §II·7 we saw that the condition for exactness of $M + Ny'$ is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. In our case

$$M = \mu P y - \mu Q \quad \text{and} \quad N = \mu.$$

Here

$$\frac{\partial M}{\partial y} = \mu P \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{d\mu}{dx},$$

since μ is a function of x only. Hence the equation will be exact if

$$\mu P = \frac{d\mu}{dx}, \quad \text{or} \quad \frac{d\mu}{\mu} = P dx.$$

Solving this, we find $\mu = ce^{\int P dx}$. For $c = 1$ we have the integrating factor $\mu = e^{\int P dx}$.

Making use of the fact that

$$\frac{d}{dx}\left[ye^{\int P dx}\right] = y'e^{\int P dx} + Pye^{\int P dx},$$

we see that, on *multiplying both members of the differential equation* (1) *by the integrating factor* $e^{\int P dx}$ *and integrating both members, one obtains the solution in the form*

$$ye^{\int P dx} = \int Qe^{\int P dx}\, dx + c.$$

Methods applicable to the solution of linear differential equations of any order are available for solving a linear differential equation of the first order.* But the method of introducing a known integrating factor is undoubtedly simplest in practice and, besides, has the advantage of being readily retained in mind.

---

* One of these methods (given in §VII·1) is the following:

Temporarily let $Q = 0$. Equation (1) then becomes

$$\frac{dy}{dx} + Py = 0, \tag{2}$$

in which the variables are separable. It is readily seen that $y = e^{-\int P dx}$ is a solution of (2). Represent it by $y = y_1$.

Now introduce a new dependent variable v by letting $y = y_1 v$. Then $\frac{dy}{dx} = y_1\frac{dv}{dx} + v\frac{dy_1}{dx}$, and equation (1) becomes

$$y_1\frac{dv}{dx} + \left(\frac{dy_1}{dx} + Py_1\right)v = Q.$$

Since y_1 is an integral of (2), this equation reduces to

$$e^{-\int P dx}\frac{dv}{dx} = Q, \quad \text{or} \quad dv = Qe^{\int P dx}dx.$$

A quadrature gives

$$v = \int Qe^{\int P dx}dx + c.$$

Finally, from $y = y_1 v$, we have

$$y = e^{-\int P dx}\int Qe^{\int P dx}dx + ce^{-\int P dx}.$$

The solution of (1) by a second general method applicable to linear differential equations of any order will be found in §VI·9.

***Exercise* 1.** $xy' + (1 + x)y = e^x$.

Divided through by the coefficient of y', our equation takes the form

$$y' + \left(\frac{1}{x} + 1\right)y = \frac{e^x}{x}.$$

Here $P = 1/x + 1$. Hence $\int P\,dx = \log x + x$, and the integrating factor is $e^{\log x + x}$, or xe^x. Using this, we have the solution in the form

$$yxe^x = \int e^{2x}dx = \tfrac{1}{2}e^{2x} + c, \quad \text{or} \quad 2xy = e^x + c_1e^{-x}.$$

2. $y' + y \cot x = \sec x$.
3. $(x^2 + 1)y' - xy = 1$.
4. $(x + 1)y' - 2y = (x + 1)^4$.
5. $(x + x^3)y' + 4x^2y = 2$.
6. $x^2y' + (1 - 2x)y = x^2$.
7. $(x^2 - 1)y' - xy = x^3 - 2x$.
8. $ydx - (3x + y^4)dy = 0$. Let x be the dependent variable.

II·10. Equation reducible to a linear equation. At times an obvious transformation will change an equation into a linear one. An equation, of frequent occurrence, for which this is true, is

$$y' + Py = Qy^k \text{*}$$

where, as before, P and Q are functions of x only, and k is any number.

Dividing by y^k, we have

$$y^{-k}y' + Py^{-k+1} = Q.$$

Let $y^{-k+1} = v$. Then $(1 - k)y^{-k}y' = v'$, and our equation becomes

$$v' + (1 - k)Pv = (1 - k)Q,$$

which is linear.

Occasionally, inspection will suggest a transformation of the dependent variable that will reduce a given equation to a linear one. This is true in the case of Exercises 2, 3, and 4 below.

---

* This is known as Bernoulli's equation, after James Bernoulli (1654–1705).

Exercise **1.** $3(1 + x^2)y' + 2xy = 2xy^4$.

Dividing by y^4, we have

$$3(1 + x^2)y^{-4}y' + 2xy^{-3} = 2x.$$

The combination $y^{-4}y'$ suggests the derivative of y^{-3}, which also appears in the equation. Putting $y^{-3} = v$, whence $-3y^{-4}y' = v'$, and dividing by $-(1 + x^2)$, we obtain

$$v' - \frac{2x}{1 + x^2}v = -\frac{2x}{1 + x^2},$$

a linear differential equation. An integrating factor is $1/(1 + x^2)$. Introducing this and integrating, we have

$$\frac{v}{1 + x^2} = -\int \frac{2x}{(1 + x^2)^2}\,dx + c.$$

Letting $1 + x^2 = u$, the integral to be evaluated becomes $-\int \frac{du}{u^2}$, which is equal to $\frac{1}{u}$, and our solution takes the form

$$\frac{y^{-3}}{1 + x^2} = \frac{1}{1 + x^2} + c, \text{ or } y^{-3} = 1 + c(1 + x^2).$$

Exercise **2.** $xy' - y \log y = x^2y$.

If we write this in the form

$$\frac{xy'}{y} - \log y = x^2,$$

the combination y'/y suggests the derivative of log y, and log y also appears in the equation. If we let $\log y = v$, the equation takes the form

$$xv' - v = x^2,$$

which is linear. The solving of this equation will be left for the student.

3. $yy' + xy^2 = x^3$. **4.** $\sin y \cdot y' + \sin x \cos y = \sin x$.

5. $4xy' + 3y + e^x x^4 y^5 = 0$. **6.** $(1 - x^3)y' - 2(1 + x)y = y^{5/2}$.

7. $(x + 1)y' = y + 1 + (x + 1)\sqrt{y + 1}$.

8. $(x^3y^3 + xy)y' = 1$. Let y be the independent variable.

9. Show that the introduction of the new variable $y^{s+1} = v$ reduces the equation $x^r y^s(mydx + nxdy) = f(x)dx$ to a linear differential equation.

10. Show that the introduction of a new variable $x^{r+1} = u$ reduces the equation $x^r y^s(mydx + nxdy) = f(y)dy$ to a linear differential equation, if y is taken as the independent variable.

II·11. Integrating factor for the homogeneous equation, §II·2, and the isobaric equation, §II·4.

1° *An integrating factor for* $Mdx + Ndy = 0$, *when* M *and* N *are homogeneous and of the same degree* r, *is*

$$\frac{1}{xM + yN}.$$

To verify this, we shall show that

$$\frac{\partial}{\partial y}\frac{M}{xM + yN} - \frac{\partial}{\partial x}\frac{N}{xM + yN} = 0. \tag{1}$$

If we actually perform the operations indicated in the left-hand member of (1), the latter is seen, after simple reductions, to take the form

$$\frac{N\left(x\dfrac{\partial M}{\partial x} + y\dfrac{\partial M}{\partial y}\right) - M\left(x\dfrac{\partial N}{\partial x} + y\dfrac{\partial N}{\partial y}\right)}{(xM + yN)^2}. \tag{2}$$

By Euler's theorem for homogeneous functions,*

$$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = rM \quad \text{and} \quad x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = rN.$$

It follows therefore that (2) vanishes. Hence, $1/(xM + yN)$ is an integrating factor.

2° We saw that the isobaric equation, studied in §II·4 (which includes the homogeneous equation for the special value $m = 1$), takes the form

$$\frac{dy}{dx} = x^{m-1}\phi\left(\frac{y}{x^m}\right)$$

---

* For example, see the author's Calculus, §113, pp. 366–368.

when solved for dy/dx. Clearing of fractions, we may write this in the form $Mdx + Ndy = 0$, where

$$M = x^{m-1}\phi\left(\frac{y}{x^m}\right) \quad \text{and} \quad N = -1.$$

An integrating factor for this equation is

$$\frac{1}{xM + myN}.$$

To verify this, one must show that

$$\frac{\partial}{\partial y}\,\frac{x^{m-1}\,\phi\left(\dfrac{y}{x^m}\right)}{x^m\phi\left(\dfrac{y}{x^m}\right) - my} = \frac{\partial}{\partial x}\,\frac{-1}{x^m\phi\left(\dfrac{y}{x^m}\right) - my}.$$

This is not difficult, and is left as an exercise.

Remark. These methods cease to apply if $xM + yN = 0$ in the first case, and if $xM + myN = 0$ in the second. But in either of these cases the solution of the equation is effected directly with ease. For, if $xM + yN = 0$, the equation takes the form $y' = -\dfrac{M}{N} = \dfrac{y}{x}$, and its solution is $y = cx$; on the other hand, when $xM + myN = 0$, the equation takes the form $y' = -\dfrac{M}{N} = \dfrac{my}{x}$, and the solution is $y = cx^m$.

Leibniz (1646–1716) and his school endeavored to solve all equations of the first order by introducing new variables which are separable in the transformed equation. Euler (1707–1783) and his school tried to solve all equations of the first order by finding integrating factors. As a matter of fact, these are frequently spoken of as *Euler factors* or *multipliers*. But the idea of an integrating factor seems to be due to a contemporary of his, Clairaut (1713–1765). Indeed, it is interesting to note that, just as, in general, every differential equation of the first order, $Mdx + Ndy = 0$, has an integrating factor, so it can be proved* that by a proper change of variables every such equation can be transformed into one in which the variables are separable. But in actual practice it would be awkward and difficult to carry out this method in all cases, just as it would be inadvisable to insist upon finding an integrating factor in every instance.

---

* For example, see the author's *Lie Theory of One-Parameter Groups*, Chap. II. In particular, see §§12, 15, 20.

The two classes of equations mentioned above* are of particular interest as affording examples of cases where both the general methods of solution can be applied readily.

Exercise. Solve the exercises of §§II·2 and II·4 by finding integrating factors for them.

★II·12. Equation of the form

$$\boldsymbol{x^r y^s(mydx + nxdy) + x^\rho y^\sigma(\mu ydx + \nu xdy) = 0.}$$

Since $d(x^a y^b) = x^{a-1}y^{b-1}(aydx + bxdy)$, the equation

$$x^r y^s(mydx + nxdy) = 0$$

is seen to have an integrating factor of the form $x^\alpha y^\beta$, where $\alpha + r = m - 1$ and $\beta + s = n - 1$.

More generally, it may be noted that *an equation of the form*

$$x^r y^s(mydx + nxdy) + x^\rho y^\sigma\,(\mu ydx + \nu xdy) = 0$$

also has an integrating factor of the form $x^\alpha y^\beta$.

Thus, multiplying by $x^\alpha y^\beta$ and rearranging the terms, we have

$$(mx^{\alpha+r}y^{\beta+s+1} + \mu x^{\alpha+\rho}y^{\beta+\sigma+1})dx \\ + (nx^{\alpha+r+1}y^{\beta+s} + \nu x^{\alpha+\rho+1}y^{\beta+\sigma})dy = 0.$$

This is exact, provided $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$; i.e.,

$$m(\beta + s + 1)x^{\alpha+r}y^{\beta+s} + \mu(\beta + \sigma + 1)x^{\alpha+\rho}y^{\beta+\sigma} \\ \equiv n(\alpha + r + 1)x^{\alpha+r}y^{\beta+s} + \nu(\alpha + \rho + 1)x^{\alpha+\rho}y^{\beta+\sigma}$$

for all values of x and y; that is, provided

$$m(\beta + s + 1) = n(\alpha + r + 1) \quad \text{and} \quad \mu(\beta + \sigma + 1) = \nu(\alpha + \rho + 1).$$

---

* While the two methods of this section can only be verified at this time, they are suggested as special cases of a general theory, due to Sophus Lie, already referred to in the previous footnote.

★ This section may be omitted in a short course.

These two linear equations can be solved for α and β, except when

$$\frac{\mu}{m} = \frac{\nu}{n} = k.$$

But, in this exceptional case, the equation reduces to

(1) $$(x^r y^s + kx^\rho y^\sigma)(my\,dx + nx\,dy) = 0,$$

or simply to

(2) $$my\,dx + nx\,dy = 0,$$

in which the variables are separable.

A *special* solution of (1) may be obtained by equating its other factor to zero, thus

$$x^r y^s + kx^\rho y^\sigma = 0.$$

While this solution is not to be ignored, it usually has no connection with the general solution obtained by solving (2).

Exercise **1.** $x^4y(3y\,dx + 2x\,dy) + x^2(4y\,dx + 3x\,dy) = 0.$

Multiplying by $x^\alpha y^\beta$ and rearranging the terms, we have

$$(3x^{\alpha+4}y^{\beta+2} + 4x^{\alpha+2}y^{\beta+1})dx + (2x^{\alpha+5}y^{\beta+1} + 3x^{\alpha+3}y^\beta)dy = 0.$$

This is exact if

$$\begin{aligned}(3\beta + 6)x^{\alpha+4}y^{\beta+1} &+ (4\beta + 4)x^{\alpha+2}y^\beta \\ &\equiv (2\alpha + 10)x^{\alpha+4}y^{\beta+1} + (3\alpha + 9)x^{\alpha+2}y^\beta,\end{aligned}$$

or

$$3\beta + 6 = 2\alpha + 10 \quad \text{and} \quad 4\beta + 4 = 3\alpha + 9;$$

i.e., if $\alpha = 1$, $\beta = 2$. An integrating factor is thus found to be xy^2. Introducing it, we have

$$(3x^5y^4dx + 2x^6y^3dy) + (4x^3y^3dx + 3x^4y^2dy) = 0.$$

Each group of terms is separately exact. The solution is

$$\tfrac{1}{2}x^6y^4 + x^4y^3 = c, \quad \text{or} \quad x^6y^4 + 2x^4y^3 = c_1.$$

2. $2y\,dx + x\,dy + xy(3y\,dx + 2x\,dy) = 0.$

3. $x^2y(y\,dx + 2x\,dy) - (y\,dx - x\,dy) = 0.$

4. $y^2(3y\,dx - 6x\,dy) - x^3(y\,dx - 2x\,dy) = 0.$

5. $2x^3(ydx - xdy) - y(ydx + xdy) = 0.$

6. $x^2y(10ydx + 9xdy) - (2ydx - 3xdy) = 0.$

★II·13. Other forms for which integrating factors can be found. In applying the test for exactness (§II·7), the value of $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}$ is obtained. If this is found to be zero, the equation is exact. If not, it may contain either M or N as a factor. In particular,

1° *if* $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = N \cdot f(x)$, *where* $f(x)$ *is a function of* x *only,* $e^{\int f(x)dx}$ *is an integrating factor;*

2° *if* $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = M \cdot \phi(y)$, *where* $\phi(y)$ is *a function of* y *only,* $e^{\int \phi(y)dy}$ *is an integrating factor.*

To establish the truth of the first statement, we shall show that the equation

$$Mdx + Ndy = 0 \tag{1}$$

has an integrating factor $\mu(x)$, which is a function of x only, when M and N satisfy the condition expressed in case 1°.

Multiplying (1) by $\mu(x)$ and expressing the condition for exactness, we have

$$\mu\frac{\partial M}{\partial y} - \mu\frac{\partial N}{\partial x} - N\frac{d\mu}{dx} = 0,$$

or

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{N}{\mu}\frac{d\mu}{dx} = N\frac{d\log\mu}{dx}.$$

Comparing this with the condition on M and N, we see that we may take

$$\frac{d\log\mu}{dx} = f(x), \quad \text{or} \quad \mu = e^{\int f(x)dx}.$$

★ This section may be omitted in a short course.

The establishing of case 2° will be left as an exercise.

***Exercise* 1.** $(3x^2 + 6xy + 3y^2 + 2y)dx + (2x^2 + 3xy + x)dy = 0.$

2. $2xydx - (x^2 + y^2 - a^2)dy = 0.$

3. $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$

II·14. Integrating factor by inspection. Integrating factors can frequently be found by inspection on closely examining the terms appearing in the equation. No general rule can be given. But certain combinations often suggest more or less obvious integrating factors.

A commonly occurring group of terms is

$$ydx - xdy.$$

By inspection, $\dfrac{1}{x^2}, \dfrac{1}{y^2}, \dfrac{1}{xy}, \dfrac{1}{x^2 + y^2}, \dfrac{1}{x^2 - y^2}$ are all readily seen to be factors which make this combination exact. For,

$$\frac{ydx - xdy}{x^2} = -d\left(\frac{y}{x}\right), \quad \frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right), \quad \frac{ydx - xdy}{xy} = d\log\left(\frac{x}{y}\right),$$

$$\frac{ydx - xdy}{x^2 + y^2} = d\tan^{-1}\frac{x}{y} = -d\tan^{-1}\frac{y}{x}, \quad \frac{ydx - xdy}{x^2 - y^2} = \tfrac{1}{2}d\log\frac{x - y}{x + y}\,.$$

Hence, $1/x^2$ is an integrating factor for an expression of the form $ydx - xdy + f(x)dx$; and $1/y^2$ may be used for one of the form $ydx - xdy + f(y)dy$; similarly, $1/xy$ may be used for an expression of the form $ydx - xdy + f(xy)(ydx + xdy)$, and so may $1/(x^2 \pm y^2)$ be used in the case of

$$ydx - xdy + f(x^2 \pm y^2)(xdx \pm ydy).$$

If $f(x, y)dx + \phi(x, y)dy$ is known to be exact, let it be represented by du. Then an equation of the form

$$f(x, y)dx + \phi(x, y)dy = F(u)\Phi(x)dx \quad [\text{or} = F(u)\Phi(y)dy]$$

has the obvious integrating factor $1/F(u)$.

Other combinations will occur to one in actual practice.

***Exercise* 1.** $(y^2 - x^2y)dx + x^3dy = 0.$

Writing this in the form

$$y^2dx - x^2(ydx - xdy) = 0,$$

we see that $1/x^2y^2$ is an integrating factor. Using it, we obtain

$$\frac{dx}{x^2} - \frac{ydx - xdy}{y^2} = 0;$$

whence

$$-\frac{1}{x} - \frac{x}{y} = c, \quad \text{or} \quad x^2 + cxy + y = 0.$$

***Exercise* 2.** $(mx + y)dx - (x - my)dy = 0.$

Writing this in the form

$$m(xdx + ydy) + ydx - xdy = 0,$$

$1/(x^2 + y^2)$ is seen to be an integrating factor.* Using it, we obtain the solution in the form

$$m \log \sqrt{x^2 + y^2} - \tan^{-1}\frac{y}{x} = c.$$

***Exercise* 3.** $xdy - ydx = x\sqrt{x^2 - y^2}dy.$

Writing this in the form

$$\frac{xdy - ydx}{\sqrt{x^2 - y^2}} \equiv \frac{xdy - ydx}{x\sqrt{1 - \left(\frac{y}{x}\right)^2}} = xdy,$$

we see that $1/x$ is an integrating factor. Using it, we find the solution to be

$$\sin^{-1}\frac{y}{x} = y + c.$$

***Exercise* 4.** $(2x^3 - y^3 - 3x)dx + 3xy^2dy = 0.$

If this is written in the form

$$(2x^3 - 3x)dx - (y^3dx - 3xy^2dy) = 0,$$

the new variable $v = y^3$ suggests itself. The second set of terms then takes the form

$$-(vdx - xdv).$$

---

* Method 1° of §II·11 suggests the integrating factor $1/m(x^2 + y^2)$.

Because of the form of the rest of the equation, $1/x^2$ is the integrating factor to be used. Introducing it, we have

$$\left(2x - \frac{3}{x}\right) dx - \frac{vdx - xdv}{x^2} = 0.$$

Integrating, we obtain

$$x^2 - 3 \log x + \frac{v}{x} = c.$$

Replacing v by its value and clearing of fractions, we have finally

$$x^3 - 3x \log x + y^3 - cx = 0.$$

5. $xdy - ydx = (x^2 + y^2)dx.$ **6.** $xdx - ydy = x\sqrt{x^2 - y^2}dx$

7. $(x - y^2)dx + 2xydy = 0.$

8. $(y - xy^3)dx + xdy = 0.$

9. $(x^2y^2 - y)dx + (2x^3y + x)dy = 0.$

10. $(x^2 - y^2 + a^2)dx + 2xydy = 0.$

II·15. Other forms for which integrating factors can be found. In addition to the types of equations, already noted, for which integrating factors have been found, others are occasionally met. The general problem of finding an integrating factor for a given differential equation leads to the solution of a linear partial differential equation of the first order (see §XIII·3). Occasionally, this general method suggests a practical way of finding an integrating factor.

II·16. Transformation of variables. In a number of cases we have already made use of the fact that the introduction of a new variable changes the form of the differential equation. As a general mode of procedure, it is advisable to find, if possible, a transformation of variables that will reduce the equation to a known type or that will simplify its form. No general rule can be formulated. In §§II·2, II·3, II·4, II·5, II·10 serviceable transformations were used. Also, in other cases, the form of the equation may suggest the transformation to be tried. The following exercises illustrate this.

Exercise **1.** $xdy - ydx = (x^2 + y)^2dx$.

The combination $xdy - ydx$ suggests the transformation $y = vx$. For then

$$xdy - ydx = x^2dv.$$

After this transformation, the differential equation becomes

$$\frac{dv}{dx} = (x + v)^2,$$

which is the type of equation considered in §II·5. Its solution will be left as an exercise.

Exercise **2.** $xdy + ydx = (x^3y^2 - x)dx$.

The combination $xdy + ydx$ suggests the transformation $v = xy$. Using this, we find that the equation takes the form

$$dv = x(v^2 - 1)dx,$$

in which the variables are separable. The finding of the solution will be left as an exercise.

Exercise **3.** $y^2(xdx + ydy) + x(ydx - xdy) = 0$.

Here $xdx + ydy$ suggests $d(x^2 + y^2)$, while $ydx - xdy$ suggests $d(y/x)$. This combination suggests polar coördinates, where

$$x^2 + y^2 = \rho^2 \quad \text{and} \quad \tan^{-1}\frac{y}{x} = \theta.$$

Differentiating these, we have

$$xdx + ydy = \rho d\rho \quad \text{and} \quad ydx - xdy = -\rho^2d\theta.$$

Hence, when the new variables ρ and θ are introduced, the differential equation takes the form

$$\rho^2 \sin^2\theta \cdot \rho d\rho - \rho\cos\theta \cdot \rho^2d\theta = 0,$$

or

$$d\rho - \frac{\cos\theta}{\sin^2\theta}d\theta = 0.$$

Integrating, we have

$$\rho + \csc\theta = c, \quad \text{or} \quad \rho\sin\theta + 1 = c\sin\theta.$$

Passing back to the original variables, we find that this becomes

$$y + 1 = \frac{cy}{\sqrt{x^2 + y^2}},$$

or, finally

$$(x^2 + y^2)(y + 1)^2 = c_1y^2.$$

***Exercise* 4.** $(x - y^2)dx + y(1 + x)dy = 0.$

Because y and dy occur only in the combinations y^2 and ydy, the new variable $y^2 = v$ is suggested.

On the other hand, writing the equation in the form

$$xdx + ydy - y(ydx - xdy) = 0$$

suggests the introduction of polar coördinates.

It is recommended that the equation be solved by the use of each of these changes of variables.

5. $(x - y^2)dx + y(1 - x)dy = 0.$

6. $xdy - ydx + x^2(2y - x)dx = 0.$

7. $(x^2 - y^2 - 1)xdx + (x^2 + y^2 - 3)ydy = 0.$

8. $y' + x = \sqrt{x^2 + y}.$ Let $x^2 + y = u^2.$

II·17. Summary. In actual practice, when the equation $Mdx + Ndy = 0$ is to be integrated, we proceed as follows:

By inspection we can tell when

1° the variables are separable (§II·1),

2° M and N are homogeneous and of the same degree (§§II·2, II·11),

3° the equation is linear or readily reducible to one that is (§§II·9, II·10),

4° M and N are linear but not homogeneous (§II·3),

5° the equation is of the form $y' = F(ax + by + c)$ (§II·5),

6° the equation is of the form (§II·12)

$$x^ry^s(mydx + nxdy) + x^\rho y^\sigma(\mu ydx + \nu xdy) = 0.$$

If, on inspection, the equation cannot be classified in any of these ways, the following possibilities should be considered:

7° The equation may be exact, i.e., $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 0$ (§II·8).

8° $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = N \cdot f(x)$ (§II·13).

9° $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = M \cdot \phi(y)$ (§II·13).

10° The equation may be isobaric (§§II·4, II·11).

11° An integrating factor may be found by inspection, or by the general method of §XIII·3 (§§II·14, II·15).

12° It may be possible to find a transformation that will reduce the equation to a known type, or simplify it (§II·16).

Additional methods for the solution of differential equations of the first order are given in Chapter IV.

***Exercise* 1.** $x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0.$

2. $\sqrt{1-y^2}dx + \sqrt{1-x^2}dy = 0.$

3. $y' - x^2y = x^5.$ **4.** $(y-x)^2y' = 1.$

5. $xy' + y + x^4y^4e^x = 0.$ **6.** $(1-x)ydx + (1-y)xdy = 0.$

7. $(y-x)dy + ydx = 0.$ **8.** $xdy - ydx = \sqrt{x^2+y^2}dx.$

9. $xy' - y = \sqrt{x^2-y^2}.$ **10.** $(x+a)y' - 3y = 2(x+a)^5.$

11. $\left(x\sin\dfrac{y}{x} - y\cos\dfrac{y}{x}\right)dx + x\cos\dfrac{y}{x}\,dy = 0.$

12. $(x-2y+5)dx + (2x-y+4)dy = 0.$

13. $(x^2+y^2)dx + 2ydy = 0.$

14. $(2x^2-1)ydx - (x^2-1)xdy = 0.$

15. $\dfrac{dy}{dx} + \dfrac{y}{(1-x^2)^{3/2}} = \dfrac{x+(1-x^2)^{1/2}}{(1-x^2)^2}.$

16. $(1-x^2)y' - xy = axy^2.$

★17. $xy^2(3ydx + xdy) - (2ydx - xdy) = 0.$

★18. $(5xy - 3y^3)dx + (3x^2 - 7xy^2)dy = 0.$

19. $ydx - x(1+xy^3)dy = 0.$ **20.** $(1+x^2)y' + y = \tan^{-1}x.$

21. $y' + y\cos x = \sin 2x.$ **22.** $(xy^2+y)dx - xdy = 0.$

23. $3x^2ydx - (x^3+y^2)dy = 0.$ **24.** $3x^2ydx + (x^3+y^2)dy = 0.$

★ This exercise may be omitted in a short course.

25. $(x^2 + y^2)(xdx + ydy) = (x^2 + y^2 + x)(xdy - ydx).$

26. $(2x + 3y - 1)dx + (2x + 3y - 5)dy = 0.$

27. $(2x^3y^2 - y)dx + (2x^2y^3 - x)dy = 0.$

28. $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0.$

29. $xdy + (y - y^2 \log x)dx = 0.$

30. $(x^2 + y^2)(xdx + ydy) + (1 + x^2 + y^2)^{1/2}(ydx - xdy) = 0.$

★**31.** $(x^4y^3 - x^3y^2 - x^2y + x)y' + (x^3y^4 + x^2y^3 + xy^2 + y) = 0.$

32. $(2xy - x)dy + ydx = 0.$ **33.** $4x^2y^2dx + (2x^3y - 1)dy=0.$

34. $(e^x + 3y^2)dx + 2xydy = 0.$ **35.** $y' + 2xy = 2ax^3y^3.$

36. $ydx + (2\sqrt{xy} - x)dy = 0.$

37. $3x - x^2 + 2y^2 - xy^2 + 2xyy' = 0.$

38. $(\sqrt{x^2 + y^2} - x)(xdx + ydy) - y(xdy - ydx) = 0.$

★ This exercise may be omitted in a short course.

CHAPTER III

APPLICATIONS

III·1. Differential equation of a family of curves. A differential equation of the first order and degree

$$(1)\qquad f(x, y, y') = 0,$$

when solved for the derivative

$$(2)\qquad y' = F(x, y),$$

is seen to determine a unique value of the derivative, corresponding to any given pair of values of the variables for which $F(x, y)$ has a definite value.*

If x and y are the rectangular coördinates of a point on a curve $y = y(x)$, the value of the derivative y' is equal to the slope of the tangent to the curve† at the point (x, y). Thus, *the differential equation of the first order and degree* (1) *determines a unique direction at each point* (x, y) *of the plane where* $F(x, y)$ *has a definite value.* Since an infinite slope corresponds to a definite direction, points at which $F(x, y)$ becomes infinite need not be excluded here.

If

$$(3)\qquad \phi(x, y, c) = 0$$

is the general solution of (1) or (2), particular solutions are obtained from it by assigning special values to c. Thus, putting $c = c_0$, a definite constant, we have the particular solution

$$(4)\qquad \phi(x, y, c_0) = 0.$$

---

* For exceptional pairs of values of x and y, $F(x, y)$ may become indeterminate. This occurs, for example, in the case of $y' = y/x$, when $x = 0$, $y = 0$.

† We shall, at times, use the briefer expression "slope of the curve" at the point (x, y).

The value of y', obtained from it, corresponding to a given pair of values of x and y satisfying (4), must be the value of y' determined by the differential equation. The curve (4) is said to be an *integral curve* of the differential equation, and is characterized by the fact that *at each of its points* (x, y) *it has the direction determined by the differential equation at that point.*

Fixing our attention upon any particular point (x_0, y_0), where $F(x_0, y_0)$ has a definite value, we can find the integral curve through it by solving the equation

$$\phi(x_0, y_0, c) = 0 \tag{5}$$

for c, and substituting this value in (3).

If the differential equation (1) is of the second degree, $F(x, y)$ will be a two-valued function, so that to each pair of values of x and y there will correspond, in general,* two values of y'; that is, the differential equation determines, in general, two directions at each point of the plane. Through such points two distinct integral curves will pass (except for points at which an integral curve has a double point). The equation (3) thus determines two values of c, in general, corresponding to a given pair of values of x and y. Hence, if $\phi(x, y, c)$ is a polynomial in c, it is of the second degree in c.

Extending this reasoning to the case of a differential equation of the first order and nth degree, we can establish the

Theorem. *The general solution* $\phi(x, y, c) = 0$ *of a differential equation of the first order and* n*th degree is of the* n*th degree in* c *if* $\phi(x, y, c)$ *is expressed as a polynomial in* c.

There is no corresponding theorem for differential equations of higher order than the first, except in the case of linear differential equations.

Another very important idea may be developed from the following considerations.

---

* Corresponding to those pairs of values of x and y for which the polynomial $f(x, y, y')$ is a perfect square, the two values of y' will coincide. The loci of such special points will be seen later (Chapter V) to play an important rôle.

The general solution (3) represents a family of curves, where to each value of c there corresponds a definite integral curve. Since any one of an indefinite number of values may be assigned to c, (3) may be said to be the equation of a family of a *single infinity of curves*, or simply, a *family of* ∞^1 *curves.* Since (3) is the primitive of (1), the latter is known as the *differential equation of the family of curves* (3). We thus see that (3) *and* (1) *are two ways of expressing the equation of a family of a single infinity of curves, the one involving an arbitrary constant, the other being a differential equation of the first order.*

In an analogous manner we shall say that

$$\phi(x, y, c_1, c_2, \ldots, c_k) = 0, \tag{6}$$

involving k arbitrary constants, is the *equation of a family of* ∞^k *curves*, and the *differential equation of this family* is the differential equation of the kth order

$$f(x, y, y', y'', \ldots, y^{(k)}) = 0, \tag{7}$$

which has (6) for its primitive.*

Starting with the form (3) [or (6)], we can always proceed to (1) [or (7)] by a direct process. A problem of frequent occurrence is that of determining the form (3) [or (6)] when (1) [or (7)] is given.

The following exercises illustrate the first type of problem mentioned here. The second type for families of ∞^1 curves will be considered in §III·2.

***Exercise* 1.** Find the differential equation of all circles through the origin with their centers on the axis of x.

The equation of this family of circles is evidently $x^2+y^2-2cx=0$. Here c enters to the first degree. Differentiating, we have

$$x + yy' - c = 0.$$

Eliminating c, we obtain

$$2xyy' + x^2 - y^2 = 0,$$

---

* Similar discussions apply to families of curves given in other systems of coördinates. In later sections, families of curves in polar coördinates will also be considered.

which is of the first degree. [As an exercise, the student should integrate this.]

***Exercise* 2.** Find the differential equation of all circles of fixed radius r, with their centers on the axis of x.

The equation of this family of circles is

$$(x - c)^2 + y^2 = r^2.$$

Here c enters to the second degree. Differentiating, we have

$$(x - c) + yy' = 0.$$

Eliminating c, we obtain

$$y^2y'^2 + y^2 = r^2,$$

which is of the second degree.

3. Find the differential equation of the system of confocal conics whose axes coincide with the axes of coördinates.

HINT. Since the distance of a focus of the conic $x^2/a^2 \pm y^2/b^2 = 1$ from the center is $\sqrt{a^2 \mp b^2}$, it is quite clear that all the conics $x^2/c + y^2/(c - \lambda) = 1$ have the same foci, no matter what c may be. Hence, this is the equation of a system of confocal conics whose foci are at the points $(\pm\sqrt{\lambda}, 0)$. Here c is the arbitrary constant to be eliminated.

4. Find the differential equation of the system of parabolas whose foci are at the origin of coördinates and whose axes coincide with the axis of x.

5. Find the differential equation of the family of straight lines tangent to the circle

$$x^2 + y^2 = r^2.$$

6. Find the differential equation of the family of straight lines, the sum of whose intercepts on the coördinate axes is a constant.

7. Find the differential equation of the family of nodal cubics

$$(y - c)^2 = 2x(x - 1)^2,$$

each curve of the family being tangent to the axis of y and having its node at the point $(1, c)$.

8. Find the differential equation of the family of nodal cubics

$$y^2 = 2x(x - c)^2,$$

each curve of the family being tangent to the axis of y at the origin and having its node at the point $(c, 0)$.

9. Find the differential equation of all the straight lines in the plane.

10. Find the differential equation of all circles tangent to the axis of y.

11. Find the differential equation of all central conics whose axes coincide with the axes of coördinates.

12. Find the differential equation of all central conics whose axes are parallel to the coördinate axes.

13. Find the differential equation of all parabolas whose axes are parallel (*a*) to the axis of x, (*b*) to the axis of y.

14. Find the differential equation of all circles of the same radius r.

15. Find the differential equation of all the circles in the plane.

III·2. Geometrical problems involving the solution of differential equations. Differential equations of the first order arise and must be solved in geometrical problems when a curve is given by properties whose analytic expression involves the derivative of one of the coördinates with respect to the other. Several exercises will illustrate this.

Exercise 1. Find the family of curves such that the tangent at each point of any one of them and the line joining that point with the origin (which we shall call the *radius vector* to that point) make an isosceles triangle with the axis of x, the latter forming the base.

The tangent of the angle which the tangent line makes with the axis of x is y', while that of the angle which the radius vector makes with the axis of x is y/x. Hence, we must have

$$y' \equiv \frac{dy}{dx} = -\frac{y}{x}, \quad \text{or} \quad xdy + ydx = 0.$$

Integrating, we have

$$xy = c,$$

which is the equation of a family of equilateral hyperbolas.

A particular solution is obtained when a definite curve of the family of integral curves is selected by some special condition it must satisfy, such as passing through a given point. A selective condition of this sort, which must be satisfied by a particular solution, is known as a *boundary condition* or an *initial condition.* Thus, that curve of the family of hyperbolas which passes through the point (1, 2) is obtained by putting $x = 1$, $y = 2$ in the equation of the family. This determines the value of c to be 2. Hence, $xy = 2$ is that curve of the family passing through the point (1, 2).

***Exercise* 2.** Find the family of curves such that the perpendicular distance from the origin to each tangent is equal to the abscissa of the point of contact.

Using formula (k) below, we find the differential equation of the family of curves to be

$$\frac{xy' - y}{\sqrt{1 + y'^2}} = x,$$

or, clearing of fractions and radicals,

$$y^2 - x^2 - 2xyy' = 0.$$

Integrating this, we find the equation of the family of curves to be

$$x^2 + y^2 - cx = 0,$$

a family of circles.

In particular, that circle of the family which passes through a given point (x_0, y_0) is readily found to be

$$x_0(x^2 + y^2) - (x_0^2 + y_0^2)x = 0.$$

From the above simple exercises, the general method of procedure may be seen:

First, we express analytically the given property of the curve family (this gives rise to a differential equation).

Then, we solve this equation.

Finally, we interpret geometrically the result obtained.

If a particular curve is to be found, its equation is obtained by finding that value of the constant of integration for which the boundary condition is satisfied.

In the Differential Calculus certain properties of curves involving differential expressions are usually studied, and their knowledge is presupposed here. For purpose of convenient reference, the following list is given:

I. RECTANGULAR COÖRDINATES.

(*a*) y' is the slope of the tangent to the curve at the point (x, y);

(*b*) $-1/y'$ is the slope of the normal at (x, y);

(*c*) $Y - y = y'(X - x)$ is the equation of the tangent at the point (x, y) on the curve, X and Y being the coördinates of any point on the tangent;

(*d*) $X - x + y'(Y - y) = 0$ is the equation of the normal at (x, y);

(*e*) $x - y/y'$ and $y - xy'$ are the intercepts of the tangent on the axes of x and y respectively;

(*f*) $x + yy'$ and $y + x/y'$ are the intercepts of the normal on the axes of x and y respectively;

(*g*) $(y/y')\sqrt{1 + y'^2}$* and $x\sqrt{1 + y'^2}$ are the lengths of the tangent from the point of contact to the axes of x and y respectively;

(*h*) $y\sqrt{1 + y'^2}$ and $(x/y')\sqrt{1 + y'^2}$ are the lengths of the normal from the point on the curve to the axes of x and y respectively;

(*i*) $\dfrac{xy' - y}{y'}\sqrt{1 + y'^2}$ is the length of the tangent intercepted by the coördinate axes;

(*j*) $\dfrac{x + yy'}{y'}\sqrt{1 + y'^2}$ is the length of the normal intercepted by the coördinate axes;

(*k*) $\dfrac{xy' - y}{\sqrt{1 + y'^2}}$ is the distance of the tangent from the origin;

(*l*) y/y' is the length of the subtangent;

(*m*) yy' is the length of the subnormal;

(*n*) $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}dx = \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2}dy$ is the element of length of arc;

(*p*) ydx or xdy is the element of area.

---

* Expressions involving a radical give the *numerical* value only. When any of these enter into an equation, the proper sign to be prefixed must be determined by the nature of the problem. Thus, see Ex. 5 below. Frequently it is not necessary to consider the algebraic sign. This is true when its identity is lost in rationalizing the resulting differential equation.

II. POLAR COÖRDINATES.

(q) $\tan\psi = \rho/\rho'$, where ρ' is $\dfrac{d\rho}{d\theta}$ and ψ is the angle which the tangent at (ρ, θ) makes with the radius vector to that point;

(r) $\tau = \theta + \psi$, where τ is the angle which the tangent makes with the initial line;

(s) $\rho\tan\psi = \rho^2/\rho'$ is the length of the polar subtangent;

(t) $\rho\cot\psi = \rho'$ is the length of the polar subnormal;

(u) $ds = \sqrt{d\rho^2 + \rho^2 d\theta^2} = \sqrt{\rho^2 + \rho'^2}\,d\theta$ is the element of length of arc;

(v) $\frac{1}{2}\rho^2 d\theta$ is the element of area;

(w) $p = \rho\sin\psi = \rho^2\dfrac{d\theta}{ds} = \dfrac{\rho^2}{\sqrt{\rho^2 + \rho'^2}}$ is the length of the perpendicular from the pole to the tangent.

***Exercise* 3.** Determine the curves such that the length of the normal (from a point on a curve to the axis of x) varies as the square of the ordinate of the point. In particular, find that curve which cuts the axis of y at a distance b from the origin.

Using (h), we find the differential equation of the family of curves to be

$$y\sqrt{1 + y'^2} = ky^2, \quad \text{or} \quad y^2(1 + y'^2 - k^2y^2) = 0.$$

Equating the factor y^2 to zero leads to the trivial special curve $y = 0$, which obviously satisfies the requirements, but is of little interest.

Using the remaining factor in which x is absent, we obtain, upon integrating,

$$\log(ky + \sqrt{k^2y^2 - 1}) = k(x + c), \quad [\text{or} \quad \cosh^{-1}ky = k(x + c)];$$

whence

$$y = \frac{1}{2k}\left[e^{k(x+c)} + \frac{1}{e^{k(x+c)}}\right], \quad \left[\text{or} \quad y = \frac{1}{k}\cosh\,[k(x + c)]\right],$$

a family of catenaries.

To find the particular curve, we must find that value of c which satisfies the equation

$$b = \frac{1}{2k}\left(e^{ck} + \frac{1}{e^{ck}}\right).$$

Solving this, we find that c may have either one of the two values

$$\frac{1}{k}\log\left(bk \pm \sqrt{b^2k^2 - 1}\right).$$

We thus see that two curves of the family satisfy the condition. This is consistent with the facts brought out in §III·1, since the differential equation is of the second degree.

Exercise **4.** Determine the curves such that the area included between an arc of any one of them, a fixed ordinate, a variable ordinate and the axis of x is proportional to the length of the corresponding arc.

Using (n) and (p), we have

$$\int_{x_0}^{x} y\,dx = k\int_{x_0}^{x}\sqrt{1 + y'^2}\,dx.$$

Differentiating both members, we have

$$y = k\sqrt{1 + y'^2}.$$

This is the same differential equation that arose in Ex. 3. Hence, the catenary has this property also.

Exercise **5.** Determine the curves such that the part of the tangent between the coördinate axes is bisected at the point of tangency.

Since, from the nature of the problem, y' is negative when x and y have the same sign, and positive when the latter have opposite signs, y/y' and x are always opposite in sign. Hence, using (g), we must put

$$\frac{y}{y'}\sqrt{1 + y'^2} = -x\sqrt{1 + y'^2}$$

in order to obtain the differential equation of the curve family.

6. Determine the curves such that the x-intercept of the normal at the point (x, y) is equal to k times x.

7. Determine the curves such that the x-intercept of the tangent at any point on a curve is equal to the y-intercept of the normal at that point.

8. Determine the curves such that the lengths of tangent and normal at any point on a curve, intercepted by the coördinate axes, are equal.

9. Determine the curves such that the angle between the tangent and normal at each point of a curve is bisected by the radius vector to that point.

10. Determine the curves whose subnormals are constant.

11. Determine the curves whose subtangent at each point is equal to the square of the abscissa at that point.

12. Determine the curves such that the perpendicular from the origin upon each tangent is equal in length to the ordinate of the point of contact.

13. Determine the curves such that the polar subtangent is constant.

14. Determine the curves such that the polar subtangent at any point on a curve is proportional to the radius vector to that point.

15. Determine the curves such that the polar subtangent at any point on a curve is proportional to the polar subnormal at that point.

16. Determine the curves such that the polar subnormal is constant.

17. Determine the curves such that the angle which the tangent at any point on a curve makes with the radius vector to that point is equal to the vectorial angle.

18. Determine the curves such that the angle which the tangent makes with the radius vector to the point of contact is twice the vectorial angle.

19. Determine the curves such that the angle which the tangent makes with the radius vector to the point of contact is equal to one half the vectorial angle.

20. Determine the curves such that the tangent at each point makes an isosceles triangle with the axis of x and the radius vector to the point of contact, the tangent forming the base.

III·3. Orthogonal trajectories. A curve which cuts every member of a family of curves according to some law is called a *trajectory* of the family. Thus, it may cut every curve at a constant angle,* in which case it is said to be an *isogonal trajec-*

---

* By *the angle between two intersecting curves* is meant the angle between their tangents at the point of intersection.

tory. If, in particular, the angle is a right angle, the trajectory is said to be *orthogonal.*

Similarly, if each curve of a second family of ∞^1 curves is an orthogonal trajectory of the given family, each family is said to be a *set of orthogonal trajectories of the other.*

Starting with the first family whose equation is

$$\phi(x, y, c) = 0, \tag{1}$$

we find the corresponding differential equation

$$f(y', x, y) = 0, \tag{2}$$

which, as we have noted before, determines y', the slope of the curve (or curves) of the family through the point (x, y). Obviously the differential equation

$$f\left(-\frac{1}{y'}, x, y\right) = 0 \tag{3}$$

determines, at each point (x, y), the same value (or set of values) for $-1/y'$ that the equation (2) does for y'; i.e., an integral curve of (3) passing through a point (x, y) has its slope at that point equal to the negative reciprocal of the slope of an integral curve of (2) passing through that point. Hence, each family of integral curves is a set of orthogonal trajectories of the other.

The general solution of (3) will then be the equation of the orthogonal trajectories of (1).

***Exercise* 1.** Find the orthogonal trajectories of a family of concentric circles.

If the common center is taken as origin, the equation of the circles is $x^2 + y^2 = c^2$, and their differential equation is $x + y\dfrac{dy}{dx} = 0$. Hence the differential equation of their orthogonal trajectories is

$$x - y\frac{dx}{dy} = 0, \quad \text{or} \quad \frac{dx}{x} - \frac{dy}{y} = 0.$$

Integrating, we have

$$y = cx,$$

a family of straight lines through the center of the circles.

***Exercise* 2.** Find the orthogonal trajectories of the family of circles through the origin, with their centers on the axis of x.

The equation of this family of circles is $x^2 + y^2 - cx = 0$. We have seen (Ex. 1, §III·1) that the corresponding differential equation is $2xy \dfrac{dy}{dx} + x^2 - y^2 = 0$. Therefore, the differential equation of the orthogonal trajectories is $2xy \dfrac{dx}{dy} - x^2 + y^2 = 0$.

As this equation is obviously derivable from the other by interchanging x and y, its solution must be $x^2 + y^2 - cy = 0$, the family of circles through the origin with their centers on the axis of y.

[The student should verify this result by actually integrating the equation.]

***Exercise* 3.** Show that a family of confocal central conics is self-orthogonal.

The equation of such a family of conics with their axes taken for axes of coördinates is (Ex. 3, §III·1) $\dfrac{x^2}{c} + \dfrac{y^2}{c - \lambda} = 1$, and its differential equation is $xyy'^2 + (x^2 - y^2 - \lambda)y' - xy = 0$.

Since this remains unaltered when we replace y' by $-1/y'$, we see that the family of curves is self-orthogonal. As a matter of fact, it is well known that a system of confocal central conics is made up of ellipses and hyperbolas such that through any point there pass one ellipse and one hyperbola, and these cut each other at right angles.

4. Prove that a family of parabolas having a common focus and a common axis is self-orthogonal.

5. Find the orthogonal trajectories of the family of circles through the two points $(a, 0)$ and $(-a, 0)$.

6. Find the orthogonal trajectories of the parabolas $y^2 = cx$.

7. Find the orthogonal trajectories of the equilateral hyperbolas $x^2 - y^2 = c$.

8. Find the orthogonal trajectories of the equilateral hyperbolas $x^2 - y^2 - cx = 0$.

9. Find the orthogonal trajectories of the similar central conics $ax^2 + by^2 = c$, where a and b are fixed constants and c is the arbitrary constant.

10. Find the orthogonal trajectories of the curves $ax^n + by^n = c$. Here $n \neq 2$.

11. Find the orthogonal trajectories of the central conics $ax^2 + y^2 - cx = 0$. Distinguish between the cases $a \neq 2$ and $a = 2$.

12. Find the equation of the family of involutes of the circle

$$x^2 + y^2 = a^2.$$

HINT. We saw (Ex. 5, §III·1) that the differential equation of the family of tangents to this circle is

$$y = xy' + a\sqrt{1 + y'^2}.$$

The involutes of the circle are the orthogonal trajectories of these tangents. Hence the differential equation of the family of involutes is

$$x + yy' = a\sqrt{1 + y'^2}.$$

To solve, pass to polar coördinates.

13. Prove that the differential equation of the family of trajectories which cut the integral curves of $f(y', x, y) = 0$ at a constant angle α is

$$f\left(\frac{y' - m}{1 + my'}, x, y\right) = 0,$$

where $m = \tan \alpha$.

In particular, show that the trajectories which cut the lines $y = cx$ at a constant angle α are the logarithmic spirals

$$\tfrac{1}{2} \log (x^2 + y^2) + k = \frac{1}{m} \tan^{-1} \frac{y}{x}, \quad \text{or} \quad \rho = ce^{\theta/m}.$$

14. Find the trajectories which cut at a constant angle α the circles through the origin with their centers on the axis of x.

15. Find the trajectories which cut at a constant angle α (other than a right angle) a system of concentric circles.

In polar coördinates, the equation of a family of curves may be written as

$$\phi(\rho, \theta, c) = 0 \tag{1'}$$

and its differential equation as

$$f(\rho', \rho, \theta) = 0. \tag{2'}$$

The latter defines ρ'(i.e., $d\rho/d\theta$) at each point (ρ, θ).

We have noted, §III·2(*q*), that, if ψ is the angle which the tangent at (ρ, θ) makes with the radius vector to that point, $\tan \psi = \rho/\rho'$. If, now, ψ_1 is the corresponding angle for the curve cutting this one at right angles at the point (ρ, θ), we have $\psi_1 = \psi \pm \pi/2$. Hence,

$$\tan \psi_1 = -\cot \psi = -1/\tan \psi.$$

Using the subscript 1 for quantities referring to the second curve, we have

$$\rho_1/\rho_1' = -\rho'/\rho, \quad \text{whence} \quad \rho' = -\rho\rho_1/\rho_1'.$$

At the point of intersection of the two curves $\rho_1 = \rho$, $\theta_1 = \theta$. Hence

$$f(-\rho^2/\rho', \rho, \theta) = 0 \tag{3'}$$

is the differential equation of the orthogonal trajectories of (1′), since the value of ρ/ρ' given by it equals that of $-\rho'/\rho$ determined by (2′).

16. Find the orthogonal trajectories of the family of lemniscates $\rho^2 = c \cos 2\theta$.

Differentiating this equation and eliminating c, we find the differential equation of the family to be $\rho'/\rho = -\tan 2\theta$. Hence the differential equation of the orthogonal trajectories is $\rho/\rho' = \tan 2\theta$, or

$$\frac{d\rho}{\rho} = \cot 2\theta d\theta.$$

Integrating, we find $\rho^2 = c \sin 2\theta$, a second family of lemniscates, whose common axis makes an angle of 45° with that of the first family.

17. Find the orthogonal trajectories of the family of cardioids $\rho = c(1 - \cos \theta)$.

18. Find the orthogonal trajectories of the family of logarithmic spirals $\rho = e^{c\theta}$.

19. Find the orthogonal trajectories of the family of curves $\rho^m \sin m\theta = c$.

In particular, assign to m successively various particular values, such as $1, 2, -1, -2, \frac{1}{2}, -\frac{1}{2}$, and interpret the results.

20. Find the orthogonal trajectories of the family of curves $1/\rho = \sin^2 \theta + c$.

21. Find the orthogonal trajectories of the family of curves $(\rho + 1/\rho) \cos \theta = c$.

III·4. Additional exercises giving rise to the solution of differential equations. Problems frequently arise in the physical sciences whose solution involves that of differential equations. As in the case of geometrical problems, the mode of procedure is

first, the analytic expression of the given facts of the problem (this gives rise to a differential equation);

then, the solving of this equation, with an interpretation of the result;

finally, the additional step of determining the value of the constant, or constants, of integration so as to satisfy the requirements of the problem, i.e., the initial or boundary conditions.

The following exercises will illustrate:

***Exercise* 1.** A particle falls vertically, acted upon by gravity only. If it has an initial velocity v_0, what will be its velocity at any given instant, and what will be its position at that instant?

The motion being rectilinear, a single coördinate, x, will be sufficient to determine the position of the body. Suppose the position of the body at the time $t = 0$ to be x_0 (this is called its initial position).

The velocity, which we shall represent by v, is dx/dt, and the acceleration, represented by a, is dv/dt. In case gravity acts alone, the acceleration is constant, and this constant is designated by $-g$. The negative sign is used here because distances, velocities, and accelerations are taken as positive when measured upward. We have now

$$\frac{dv}{dt} = -g.$$

Integrating this equation, we obtain

$$v = -gt + c.$$

Since $v = v_0$ when $t = 0$, it follows that $c = v_0$, and the answer to our first question is given by

$$v = -gt + v_0.$$

To find the position of the body at any instant we must integrate

$$\frac{dx}{dt} = -gt + v_0.$$

The general solution of this is

$$x = -\tfrac{1}{2}gt^2 + v_0t + c.$$

Since $x = x_0$ when $t = 0$, we must have $c = x_0$. Hence at any instant t,

$$x = -\tfrac{1}{2}gt^2 + v_0t + x_0,$$

and the distance of the particle from the starting point at any time t is

$$x - x_0 = -\tfrac{1}{2}gt^2 + v_0t.$$

2. A particle descends a smooth plane making an angle α with the horizontal plane. The only force acting is gravity. If the particle starts from rest, find the velocity at any moment t, and the distance traveled in the time t. [HINT. Here the acceleration is $g \sin \alpha$, the component of g in the direction of the motion.] Prove that, if a particle starts at rest from the highest point of a vertical circle, it will reach any other point of the circle when moving along the chord to that point in the same time that it takes to drop to the lowest point of the circle. — TAIT AND STEELE, *Dynamics of a Particle.*

3. A particle falls from rest through a resisting medium (such as air) in which the resistance is proportional to the square of the velocity; what is its motion?

The differential equation of motion is $\dfrac{dv}{dt} = k^2v^2 - g.$

Here k^2 is positive, since the resisting force is in the opposite direction from that of the falling* body. Separating the variables, we have

$$\frac{dv}{k^2v^2 - g} = dt, \quad \text{or} \quad \frac{gdv}{r^2v^2 - g^2} = dt,$$

---

* If the body is given an initial velocity upward, a separate treatment of the problem during the period of rising must be made. For example, see the author's Calculus, §80.

where $gk^2 = r^2$. Integrating, we have, after simple reductions,

$$\frac{rv - g}{rv + g} = ce^{2rt}.$$

Since $v = 0$ when $t = 0$, it follows that $c = -1$; whence

$$\frac{rv - g}{rv + g} = -e^{2rt}.$$

Solving for v, we have

$$v = \frac{dx}{dt} = \frac{g}{r}\,\frac{1 - e^{2rt}}{1 + e^{2rt}} = -\frac{g}{r}\,\frac{e^{rt} - e^{-rt}}{e^{rt} + e^{-rt}}.$$

$$\therefore x = -\frac{g}{r^2}\log\left(e^{rt} + e^{-rt}\right) + c.$$

To determine c, we make use of the fact that $x = x_0$ when $t = 0$. Hence

$$x_0 = -\frac{g}{r^2}\log 2 + c;$$

and

$$x_0 - x = \frac{g}{r^2}\log\frac{e^{rt} + e^{-rt}}{2} = \frac{g}{r^2}\log\cosh rt$$

is the distance the particle falls in the time t.

4. If a particle sinks slowly in a liquid, the resistance is proportional to the velocity. If the particle starts from rest, find the distance it falls in an interval of time t.

5. The acceleration of a particle moving in a straight line is proportional to the cube of the velocity and in the opposite direction from the latter. Find the distance passed over in the time t, the initial velocity being v_0, and the distance being measured from the initial position of the particle, i.e., $x_0 = 0$.

The differential equation of motion is

$$\frac{dv}{dt} = -k^2v^3, \quad \text{or} \quad \frac{dv}{-v^3} = k^2dt.$$

$$\therefore \frac{1}{v^2} - \frac{1}{v_0{}^2} = 2k^2t, \quad \text{or} \quad v = \frac{v_0}{\sqrt{2k^2v_0{}^2t + 1}} = \frac{dx}{dt};$$

and

$$x = \frac{\sqrt{2k^2v_0{}^2t + 1} - 1}{k^2v_0}$$

6. Find the distance passed over in the time t, if the acceleration is proportional to the velocity. Consider both cases where the acceleration and velocity have the same sign and where they have opposite signs.

7. One of the important differential equations in the Theory of Electricity arises in the case of an electric circuit with constant total resistance R, containing a source producing an electromotive force E and a coil of constant self-inductance L. As soon as the circuit is closed, the current begins to flow. If i is the value of the current at any time t ($i = 0$ when $t = 0$), the induced electromotive force is Ldi/dt. This opposes E. The net electromotive force is thus $E - Ldi/dt$. Hence

$$E - L\frac{di}{dt} = Ri,$$

or

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L},$$

which is the differential equation determining i as a function of t.

Find i (*a*) if E is constant, (*b*) if $E = E_0 \sin \omega t$, where E_0 and ω are constants, (*c*) if E is a given function of the time, viz. $E(t)$, (*d*) if, after an interval t_0 when the current is i_0, the source of the electromotive force is removed, so that from then on $E = 0$.

Case (*b*) plays such an important rôle in the Theory of Electricity that its solution is given here in detail. This equation arises in the case of alternating currents, where the electromotive force is a periodic function of the time, the period being $2\pi/\omega$, and the maximum value of the electromotive force is E_0.

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L}\sin \omega t.$$

An integrating factor is $e^{Rt/L}$.

$$\therefore\ ie^{Rt/L} = \frac{E_0}{L}\int e^{Rt/L}\sin \omega t dt + c.$$

Since $\displaystyle\int e^{at}\sin \omega t dt = \frac{a \sin \omega t - \omega \cos \omega t}{a^2 + \omega^2}e^{at},$

$$ie^{Rt/L} = \frac{E_0}{L}\,\frac{\frac{R}{L}\sin \omega t - \omega \cos \omega t}{\frac{R^2}{L^2} + \omega^2}\,e^{Rt/L} + c,$$

or

$$
\begin{aligned}
ie^{Rt/L} &= E_0\left(\frac{R\sin\omega t - \omega L\cos\omega t}{R^2+\omega^2L^2}\right)e^{Rt/L} + c \\
&= \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\,\frac{R\sin\omega t - \omega L\cos\omega t}{\sqrt{R^2+\omega^2L^2}}\,e^{Rt/L} + c \\
&= \frac{E_0e^{Rt/L}}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t-\phi) + c,
\end{aligned}
$$

where

$$\sin\phi = \frac{\omega L}{\sqrt{R^2+\omega^2L^2}},\ \cos\phi = \frac{R}{\sqrt{R^2+\omega^2L^2}}.$$

$$\therefore i = \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t-\phi) + ce^{-Rt/L}.$$

Since $i = 0$ when $t = 0$, it follows that

$$c = \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin\phi;$$

whence

$$i = \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\left[\sin(\omega t-\phi) + \sin\phi\, e^{-Rt/L}\right].$$

The term $ce^{-Rt/L}$ usually becomes negligible after a very short interval of time. The current then becomes periodic with the same frequency as the electromotive force. But the two are not in the same phase, the current lagging behind by the angle ϕ.

If, now, we impose the more general initial or boundary condition that

$$i = 0 \quad \text{when} \quad t = t_0,$$

the value of c in the expression for i may be found as follows:

Putting $t = t_0$ in this expression, we have

$$0 = \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t_0-\phi) + ce^{-Rt_0/L};$$

whence

$$c = -\frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t_0-\phi)e^{Rt_0/L}.$$

Putting this in the expression for i, we have

$$i = \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t-\phi) - \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t_0-\phi)e^{-R(t-t_0)/L},$$

or

$$i = \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} [\sin(\omega t - \phi) - \sin(\omega t_0 - \phi) e^{-R(t-t_0)/L}].$$

Another desirable form of the solution may be obtained by replacing ϕ by its value in terms of ω, L and R. From the values of $\sin \phi$ and $\cos \phi$ given above, we have

$$\tan \phi = \frac{\omega L}{R}; \quad \text{whence} \quad \phi = \tan^{-1} \frac{\omega L}{R}.$$

Using this, we find that the solution takes the form

$$i = \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \left[\sin\left(\omega t - \tan^{-1} \frac{\omega L}{R} \right) - \sin\left(\omega t_0 - \tan^{-1} \frac{\omega L}{R} \right) e^{-R(t-t_0)/L} \right].$$

It will be left, as a simple exercise, to show that, if $t_0 = 0$, the expression for i may be put in the form

$$i = \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \left[\sin\left(\omega t - \tan^{-1} \frac{\omega L}{R} \right) + \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} e^{-Rt/L} \right].$$

8. The rate at which heat is lost by a warm body is approximately proportional to the difference in temperature between the body and the surrounding medium. If a thermometer registers θ_0 degrees in a warm room and is carried into a cooler one where the temperature is θ_1 degrees, the differential equation to determine θ, the thermometer reading, is

$$\frac{d\theta}{dt} = -k^2(\theta - \theta_1).$$

What will be the reading of the thermometer at the end of an interval of time t after entering the cooler room?

If the thermometer is taken from a room in which the temperature is 64° into the open where the temperature is 0°, and at the end of 1 minute reads 32°, how long will it take until it reads 20°? 10°? 1°?

Solving the differential equation, we find, after making use of the initial condition,

$$\theta - \theta_1 = (\theta_0 - \theta_1) e^{-k^2 t}.$$

The value of $-k^2$, the factor of proportionality, may be found as follows:

In our exercise $\theta_0 = 64$, $\theta_1 = 0$. Hence, $\theta = 64\,e^{-k^2t}$. Since $\theta = 32$ when $t = 1$, we have $e^{-k^2} = \frac{1}{2}$, or $-k^2 = \log(\frac{1}{2}) = -\log 2$. This logarithm is of the Napierian system, as are most of the logarithms used in the Calculus. The solution may thus be written

$$\theta = 64\,(\tfrac{1}{2})^t, \quad \text{or} \quad \theta = 64 \cdot 2^{-t}.$$

To determine the value of t when θ assumes a given value, e.g. θ_2, we must solve

$$\theta_2 = 64 \cdot 2^{-t}$$

for t. This is effected most readily by taking the common (or Briggs) logarithm of both sides; thus

$$\text{Log}\,\theta_2 = \text{Log}\,64 - t\,\text{Log}\,2; \quad \text{whence} \quad t = \frac{\text{Log}\,64 - \text{Log}\,\theta_2}{\text{Log}\,2}.$$

9. The rate of decomposition of radium at any instant t is proportional to the amount of radium remaining at the time t. If half the original quantity disappears in 1500 years, how much remains at the end of 1000 years?

If x is the amount remaining at any time t, the law of decomposition is

$$\frac{dx}{dt} = -k^2x,$$

where $-k^2$ is a constant, essentially negative, because the quantity of radium is always diminishing.

10. In some chemical reactions, the rate of conversion of a substance is found to be proportional to the quantity of the substance still untransformed at that instant. If x is the quantity of the substance that has been converted in the interval t, and a the quantity of the substance at the beginning, show that

$$x = a(1 - e^{-kt}),$$

where k is the constant factor of proportionality.

11. Find an expression for the quantity of the substance converted during an interval t if the rate of conversion at any instant is proportional to the square of the quantity of the substance untransformed at that instant.

12. Find an expression for the quantity of the substance converted during an interval t if the rate of conversion at any instant is proportional to the cube of the quantity of the substance untransformed at that instant.

13. In a chemical reaction a substance A decomposes into two substances B and C. The rate at which each of the latter is formed at any instant t is proportional to the amount of A present at that time, the factors of proportionality being different, in general. If the amount of A is equal to a when $t = 0$, and if the amounts of B and C, at the instant t, are x and y respectively, the latter must satisfy the differential equations

$$\frac{dx}{dt} = h(a - x - y) \quad \text{and} \quad \frac{dy}{dt} = k(a - x - y).$$

Solve these and, in particular, find the values of x and y as functions of t subject to the conditions that

$$x = 0, \quad y = 0 \text{ when } t = 0, \quad \text{and} \quad x = \frac{a}{6}, \quad y = \frac{a}{3} \quad \text{when } t = 1 \text{ hour.}$$

Here we have a system of differential equations. But their solution can be effected by the successive solution of several single differential equations, each involving a single dependent variable. Thus, dividing the second equation by the first, we obtain

$$\frac{dy}{dx} = \frac{k}{h},$$

a differential equation involving x and y only. Its solution is

$$hy = kx + c.$$

Using the conditions for $t = 0$ and $t = 1$, we find $c = 0$ and $k = 2h$. Hence,

$$y = 2x.$$

Substituting this in the first differential equation of the system, we obtain

$$\frac{dx}{dt} = h(a - 3x).$$

Its solution may be written in the form

$$a - 3x = ce^{-3ht}.$$

Using the conditions for $t = 0$ and $t = 1$, we find $c = a$ and $e^{-3k} = \frac{1}{2}$. Hence,

$$a - 3x = a \cdot (\tfrac{1}{2})^t = a \cdot 2^{-t}.$$

$$\therefore x = \frac{a}{3}(1 - 2^{-t}), \quad \text{and} \quad y = 2x = \frac{2a}{3}(1 - 2^{-t}).$$

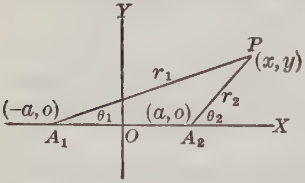
14. If two chemicals A and B react upon each other to form a compound C, a common law is that the rate at which C is formed is proportional to the product of the amounts of A and B which are still unconverted. Suppose that each molecule of C contains m molecules of A and n molecules of B. If the original number of molecules of A and B was a and b respectively, at the instant t there are $a - mx$ unconverted molecules of A and $b - nx$ of B. Show that, if $bm - an \neq 0$,

$$x = \frac{ab[e^{(bm-an)kt} - 1]}{bme^{(bm-an)kt} - an}.$$

Consider also the case where $bm - an = 0$.

15. Show that the surface of a liquid, rotating with constant angular velocity about a vertical axis, assumes the shape of a paraboloid of revolution. [There are two forces acting on each particle of the liquid, the centrifugal force and gravity; their resultant must lie in the normal to the surface.]

16. Show that the surface of a mirror which will reflect all the light coming from a fixed point along lines parallel to a fixed line is a paraboloid of revolution. [Take the fixed point as the origin, and the line through the origin and parallel to the fixed line as the axis of x.]



17. Find the equations of the lines of force induced by two magnetic poles of equal intensities m, at a distance $2a$ apart,

(a) if the poles are of like sign;
(b) if the poles are of opposite sign.

Consider first case (a).

If the poles are located at two points, A_1 and A_2, take the line joining these as the axis of x and their mid-point as the origin, as indicated in the figure.

The coördinates of A_1 and A_2 are $(-a, 0)$ and $(a, 0)$ respectively.

The magnetic force due to A_1 on a unit pole at (x, y) is equal to m/r_1^2. Its component in the direction of the axis of x is equal to

$$\frac{m}{r_1^2}\cos\theta_1, \quad \text{or} \quad \frac{m(x+a)}{r_1^3}.$$

Its component in the direction of the axis of y is equal to

$$\frac{m}{r_1^2}\sin\theta_1, \quad \text{or} \quad \frac{my}{r_1^3}.$$

Similarly, the components of the magnetic force due to A_2 on the unit pole at (x, y) are

$$\frac{m(x-a)}{r_2^3} \quad \text{and} \quad \frac{my}{r_2^3}$$

respectively. Hence the x- and y-components of the total force due to both poles are

$$m\left[\frac{x+a}{r_1^3}+\frac{x-a}{r_2^3}\right] \quad \text{and} \quad my\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]$$

respectively. The slope of the direction of this force is

$$\frac{y\left(\dfrac{1}{r_1^3}+\dfrac{1}{r_2^3}\right)}{\dfrac{x+a}{r_1^3}+\dfrac{x-a}{r_2^3}}, \quad \text{or} \quad \frac{y(r_1^3+r_2^3)}{(x+a)r_2^3+(x-a)r_1^3}.$$

Hence the differential equation of the lines of force is

$$\frac{dy}{dx}=\frac{y(r_1^3+r_2^3)}{(x+a)r_2^3+(x-a)r_1^3},$$

or

$$(x+a)r_2^3dy-yr_2^3dx+(x-a)r_1^3dy-yr_1^3dx=0,$$

or

$$\frac{(x+a)dy-ydx}{r_1^3}+\frac{(x-a)dy-ydx}{r_2^3}=0.$$

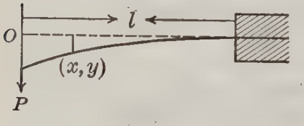
Since $r_1^2=(x+a)^2+y^2$ and $r_2^2=(x-a)^2+y^2$, it is not difficult to see, and it will be left as an exercise to show, that the solution of this differential equation is

$$\frac{x+a}{r_1}+\frac{x-a}{r_2}=c, \quad \text{or} \quad \cos\theta_1+\cos\theta_2=c.$$

18. A cantilever beam (that is, one securely fastened at one end and free at the other) is subjected to a load of P pounds, suspended from the free end. The beam is bent slightly, the length of the upper surface of the beam is slightly elongated, that of the lower surface is slightly shortened. Somewhere between these there is a section of the beam, originally plane, now a cylindrical surface, whose length has not been changed. This is known as the *neutral* surface of the beam. The curvature at any point on the section of this neutral surface by a plane normal to its elements is given by the formula*

$$\text{curvature} \equiv \frac{y''}{(1+y'^2)^{3/2}} = \frac{M}{EI},$$

where M is the bending moment at that point, E is the modulus of elasticity of the material of which the beam is made, and I is the moment of inertia of the beam. Since the deflection is slight, y' is very small and its square is much smaller. Hence, the differential equation of the plane section of the neutral surface may be taken to be



$$EIy'' = M,$$

as a first approximation.

If the length of the beam is l, and the free end extends to the left, find the amount of the deflection y of a point at a distance x from the free end. Also find the greatest deflection, that is, for $x = 0$, and the value of y' at that point. (Actually there is no point on the neutral surface of the deflected beam whose $x = 0$. But the deviation from this for the points on the left-hand boundary of the neutral surface is negligible.) Here the axis of x is the line of intersection of a vertical plane with the neutral surface before deflection, and the origin is at the free end, also before deflection. See the accompanying figure.

We shall suppose that E and I are constants, and shall neglect the weight of the beam.

From first principles we have that the bending moment of P, with respect to a point at a distance x from it, is

$$M = -Px,$$

---

* For example, see J. E. Boyd, *Strength of Materials*, Third Edition, §70.

where the − sign is used because we measure distances positively upward. The concavity of our curve on the neutral surface is downward. Hence, y'' is negative. Our differential equation is

$$EIy'' = -Px.$$

Integrating, we find

$$EIy' = -\frac{Px^2}{2} + c.$$

Since $y' = 0$ when $x = l$, we have $c = Pl^2/2$. Hence,

$$EIy' = -\frac{Px^2}{2} + \frac{Pl^2}{2}.$$

Integrating again, we find

$$EIy = -\frac{Px^3}{6} + \frac{Pl^2x}{2} + c.$$

Since $y = 0$ when $x = l$, we have $c = Pl^3/6 - Pl^3/2 = -Pl^3/3$. Hence,

$$y = -\frac{P}{6EI}(x^3 - 3l^2x + 2l^3).$$

When $x = 0$, the greatest deflection is

$$y_0 = -\frac{Pl^3}{3EI}.$$

When $x = 0$, we find the greatest value of y' to be

$$y_0' = \frac{Pl^2}{2EI}.$$

19. If, in the previous exercise, the free end extends to the right, take the fixed end as the origin of coördinates, and with this system find y and y' at a distance x along the beam; also find y and y' for $x = l$.

Here

$$M = -P(l - x).$$

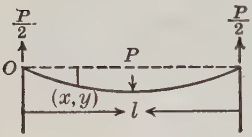
20. If the load is uniformly distributed along the beam of Ex. 18, the weight per unit length being w, find y and y' for any x, and for $x = 0$.

Here the bending moment of the part of the load extending from the free end to the point at a distance x from this end, with respect to this point, is equal to $-wx \cdot x/2$, or $-wx^2/2$. Hence the differential equation is

$$EIy'' = -\frac{wx^2}{2}.$$

The initial or boundary conditions are the same as in Ex. 18.

21. If the load is uniformly distributed along the beam of Ex. 19, the weight per unit length being w, find y and y' for any x, and for $x = l$.



Here the moment of the part of the load from the free end to the point at a distance x from the fixed end, with respect to this point, is equal to $-w(l - x)^2/2$.

22. The same differential equation

$$EIy'' = M$$

holds for a beam supported at both ends. Consider the case where a load of P pounds is suspended from the middle point of the beam. Find the value of y at $x = l/2$, and that of y' at $x = 0$.

The initial or boundary conditions are

$$y = 0 \quad \text{when} \quad x = 0, \quad \text{and} \quad y' = 0 \quad \text{when} \quad x = l/2.$$

The forces acting here are the load $-P$ at $x = l/2$, and the reactions of the supports at each end, each being equal to $P/2$. For the portion of the beam from the left end where $x = 0$ to a point at a distance x from this end, the only force to be taken into account in determining the bending moment, with respect to this point, is the reaction $P/2$ at the left-hand end. Hence, in this case, $M = Px/2$. (This applies only to the left half of the beam, that is, for $0 \leq x \leq l/2$. For the right half of the beam, x must be replaced by $l - x$.)

23. In Ex. 22, suppose the load to be uniformly distributed along the beam, the weight per unit length being w.

Here the forces, which must be taken into account in calculating the bending moment (with respect to its right-hand end) of the portion of the beam x in length, beginning at the left-hand end of the beam, are

the resistance $wl/2$ pushing vertically upward at $x = 0$,

the weight of the portion of the beam, wx, directed downward at the point $x/2$ along the beam. Hence,

$$M = \frac{wlx}{2} - \frac{wx^2}{2}.$$

The initial or boundary conditions are the same as in Ex. 22.

24. In the case of the barrel of a gun, which is a hollow cylinder, the tensile stress f at a point of the barrel, at a distance r from the axis of the cylinder, is given by John Perry in his *Calculus for Engineers*, §56, as equal to $p - 2A$, where p is the pressure in the radial direction at the point due to the explosion of the powder. A is a constant to be determined by the conditions of the experiment, and p is determined by the differential equation

$$r\frac{dp}{dr} + 2p = 2A.$$

If the radius of the inner surface of the barrel is r_0 and that of the outer surface is r_1, and if the pressure on the inner surface due to the explosion is p_0 (and, neglecting the atmospheric pressure, the pressure on the outer surface is taken as zero), find f as a function of r. What is its greatest value?

It will be left as an exercise to show that the general solution of the differential equation, determining p, is

$$pr^2 = Ar^2 + B.$$

To determine the values of A and B we use the boundary conditions

$$p = p_0 \quad \text{when} \quad r = r_0 \quad \text{and} \quad p = 0 \quad \text{when} \quad r = r_1,$$

i.e.,

$$p_0r_0^2 = Ar_0^2 + B \quad \text{and} \quad 0 = Ar_1^2 + B;$$

whence

$$A = -\frac{p_0r_0^2}{r_1^2 - r_0^2} \quad \text{and} \quad B = \frac{p_0r_0^2r_1^2}{r_1^2 - r_0^2}.$$

$$\therefore p = -\frac{p_0r_0^2}{r_1^2 - r_0^2} + \frac{p_0r_0^2r_1^2}{(r_1^2 - r_0^2)r^2}, \quad \text{and}$$

$$f = \frac{p_0r_0^2}{r_1^2 - r_0^2} + \frac{p_0r_0^2r_1^2}{(r_1^2 - r_0^2)r^2} = \frac{p_0r_0^2}{r_1^2 - r_0^2}\cdot\frac{r_1^2 + r^2}{r^2}.$$

The greatest value of f is found at $r = r_0$. Its value is

$$f_0 = p_0\frac{r_1^2 + r_0^2}{r_1^2 - r_0^2}.$$

CHAPTER IV

ADDITIONAL METHODS OF SOLUTION FOR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

IV·1. Equation solvable for y'. We shall now consider the problem of solving the differential equation

$$f(x, y, y') = 0, \tag{1}$$

where $f(x, y, y')$ is a polynomial of higher degree than the first in y', whose coefficients are rational in x and y. Equation (1) is said to be *reducible*, if it is possible to replace $f(x, y, y')$ by a product of factors, rational in x and y, each of lower degree in y'.

If the polynomial in y' cannot be so factored, (1) is said to be *irreducible.*

Suppose, now, that (1) is expressible in the form

$$f(x, y, y') = f_1(x, y, y')f_2(x, y, y') \cdots f_k(x, y, y') = 0, \tag{2}$$

where each of the k equations

$$f_i(x, y, y') = 0 \quad (i = 1, 2, \cdots, k) \tag{3}$$

is irreducible. Of course, k may be equal to 1, which is the case when the original equation is irreducible. We shall suppose that all the k equations (3) are distinct.*

The vanishing of $f(x, y, y')$ is due to the vanishing of some one of its factors. Hence the problem of solving equation (1) can be replaced by that of solving each of the equations (3). Let the solutions of the latter be

$$\phi_i(x, y, c) = 0 \quad (i = 1, 2, \cdots, k). \tag{4}$$

---

* If $f(x, y, y')$ has some of its factors repeated, there is no loss in replacing it, in (2), by the product of its distinct irreducible factors only. For, the differential equation $[f_i(x, y, y')]^r = 0$, where r is a positive integer, defines precisely the same relation between x, y, and y' as $f_i(x, y, y') = 0$, and has the same general solution as the latter.

This set of k solutions may be taken as the general solution of (1).

If desired, the general solution of (1) can be written in the form

$$\phi_1(x, y, c)\,\phi_2(x, y, c) \cdots \phi_k(x, y, c) = 0. \tag{5}$$

For, the vanishing of the left-hand member of (5) means the vanishing of some one of its factors; moreover, the equating of such a factor to zero gives a solution of one of the equations (3), and hence a solution of (1). But, in general, the k relations (4) are distinct, and it is better to treat them as such.

Consider, now,

$$f_i(x, y, y') = 0, \tag{3'}$$

one of the irreducible equations (3). Let it be of degree r, which may be 1. If r is greater than 1, $f_i(x, y, y')$ may still be replaced by a product of r factors, each of the first degree in y', but no longer rational in both y and x. Thus (3′) may be replaced by

$$[y' - F_1(x, y)]\,[y' - F_2(x, y)] \cdots [y' - F_r(x, y)] = 0. \tag{6}$$

The knowledge of any one of the factors enables one to write all the others, because $f_i(x, y, y')$ is rational in x and y. Equation (6) is equivalent to the r equations obtained by equating each of the factors of its left-hand member to zero,

$$y' - F_j(x, y) = 0 \quad (j = 1, 2, \cdots, r). \tag{7}$$

If the solution of any one of (7)

$$\Phi_j(x, y, c) = 0 \tag{8}$$

is algebraic in x and y, *the general solution of* (6) *may usually be obtained by rationalizing* (8). This follows from the fact that $F_1(x, y), F_2(x, y), \cdots, F_r(x, y)$ are the roots of an irreducible equation.

If (8) is not algebraic, the general solution of (6) may usually also be obtained from (8). Thus, see Ex. 3 below.

***Exercise* 1.** $y'^2 + (x + y)y' + xy = 0.$

This equation is reducible and may be replaced by the two equations

$$y' + x = 0 \quad \text{and} \quad y' + y = 0,$$

whose solutions are

$$2y + x^2 = c \quad \text{and} \quad y = ce^{-x},$$

respectively. These constitute the general solution of the given differential equation.

If desired, the general solution may also be put in the form

$$(2y + x^2 - c)(y - ce^{-x}) = 0.$$

***Exercise* 2.** $xy'^2 - 2yy' + x = 0.$

This is irreducible, but may be replaced by the two equations

$$xy' - y = \sqrt{y^2 - x^2} \quad \text{and} \quad xy' - y = -\sqrt{y^2 - x^2}.$$

Fixing our attention on the first of these, we recognize it as a homogeneous equation. Its solution is easily found to be

$$\frac{y}{x} + \sqrt{\frac{y^2}{x^2} - 1} = cx.$$

Rationalizing and clearing of fractions, we obtain

$$c^2x^2 - 2cy + 1 = 0,$$

which is the solution of the given differential equation. Letting $c = 1/k$, the solution may also be put in the form

$$x^2 - 2ky + k^2 = 0.$$

***Exercise* 3.** $y^2 + y'^2 = 1.$

This equation is irreducible, but may be replaced by the two equations

$$y' = \sqrt{1 - y^2} \quad \text{and} \quad y' = -\sqrt{1 - y^2}.$$

The solution of the first of these is

$$\sin^{-1}y = x + c, \quad \text{or} \quad y = \sin(x + c).$$

Either of these forms is the general solution of the given differential equation. For, the solution of the second auxiliary equation is

$$\cos^{-1}y = x + k, \quad \text{or} \quad y = \cos(x + k).$$

These become identical with the previous relations when $k = c - \pi/2$.

4. $xyy'^2 + (x^2 - y^2)y' - xy = 0$.

5. $(2xy' - y)^2 = 8x^3$. **6.** $(1 + x^2)y'^2 = 1$.

7. $y'^3 - (2x + y^2)y'^2 + (x^2 - y^2 + 2xy^2)y' - (x^2 - y^2)y^2 = 0$.

8. $y^2y'^2 - 2xyy' + 2y^2 - x^2 = 0$.

9. $(4y^2 - x^2)y'^2 - 6xyy' + 4x^2 - y^2 = 0$.

IV·2. Equation solvable for y. If the equation can be solved for y,* the following method will frequently be found useful.

Solving for y, we have

$$y = \psi(x, y'). \tag{1}$$

Differentiating this, we obtain

$$y' = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y'}\frac{dy'}{dx}, \tag{2}$$

a differential equation which is really of the second order, but, since y is no longer present, it may be looked upon as an equation of the first order in the variables x and y'. It may happen that we can integrate this equation. Suppose its solution to be

$$\omega(x, y', c) = 0. \tag{3}$$

Eliminating y' between (1) and (3), we have

$$\phi(x, y, c) = 0, \tag{4}$$

which is a solution of (1); and, since it involves an arbitrary constant, it is the general solution.

We know that (4) is a solution of (1) from the following considerations:

Since (2) was obtained by differentiating (1), every solution of (1) is a solution of (2), considered as an equation of the second order in x and y. Since (3) is a solution of (2), every solution of (3) looked upon as a differential equation in x and y is a solution of (2). Thus, the general solution of (3), since it contains a second arbitrary constant besides c, is the general solution of (2), and must, therefore, contain the solutions of (1). We proceed now to find the latter, by selecting those solutions of (3) which satisfy (1). Equation (1) has an infinite number of integral curves,* and so has (3) for each value of c. To find the curve, or curves, common to these two families of integral curves we shall have to find the equation of the locus of those points for which (1) and (3) determine the same value of y'.† But this locus is evidently obtained by eliminating y' between (1) and (3). Thus, we obtain one integral curve of (1) for each value of c. Hence, when c is an arbitrary constant, we have the general solution.

---

* It is not actually necessary to solve for y. So long as the equation (1) is linear in y, with the coefficient of y a constant, the method of this section may be employed.

At times it is easy to integrate (3). Doing this, we obtain

$$\Phi(x, y, c, c') = 0, \tag{5}$$

which, as noted above, must reduce to the general solution of (1) for some relation between c and c'. This relation may be found by substituting (5) in (1) and noting what condition is imposed in order to satisfy the equation. In actual practice, however, this method will generally not be as desirable as the one given above.

Exercise 1. $4xy'^2 + 2xy' - y = 0.$

This is linear in y, with the coefficient of y a constant. Differentiating, we have, after collecting terms and factoring,

$$(4y' + 1)\left(2x\frac{dy'}{dx} + y'\right) = 0.$$

Neglecting the factor $4y' + 1$, which does not involve the derivative dy'/dx, and whose significance we shall see later (§V·3, Note), we observe that the resulting differential equation

$$2xdy' + y'dx = 0$$

---

* We use this geometrical mode of expression, not because it is essential to the argument, but because it helps to fix the idea.

† As already mentioned in §III·1, a differential equation of the first order may be looked upon as determining y', the slope of the integral curve, at each point (x, y).

has its variables separable. Integrating, we find

$$xy'^2 = c, \quad \text{or} \quad y' = \sqrt{c/x}.$$

Substituting this in the original differential equation, we have, on rationalizing,

$$(y - 4c)^2 = 4cx.$$

Letting $4c = k$, we have finally

$$(y - k)^2 = kx.$$

Exercise **2.** $4xy' - 2y + \log y' = 0.$

Differentiating and clearing of fractions, we have

$$2y'^2dx + 4xy'dy' + dy' = 0.$$

This equation is exact, and has for solution

$$2xy'^2 + y' = c.$$

Eliminating y' between this and the original differential equation, we obtain the general solution. However, while it is possible to perform this elimination, the result will not be attractive in form. It is simpler to solve these two equations for x and y as functions of y', thus

$$x = \frac{c - y'}{2y'^2}, \quad y = \frac{c - y'}{y'} + \frac{\log y'}{2}.$$

Here our solution is given in parametric form, in terms of the parameter y'.

Such parametric representation is frequently employed. Thus, for example, we have the parametric equations of the ellipse

$$x = a \cos \theta, \quad y = b \sin \theta,$$

where θ is the eccentric angle, and, again, we have the parametric equations of the cycloid

$$x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta).$$

3. $xy'^2 - 2yy' - x = 0.$

4. $8y'^3 + 12y'^2 = 27(x + y).$

5. $y + xy' = x^4y'^2.$

6. $y'^2 + 2xy' - y = 0.$

7. $2x^3y - x^4y' + 3y'^3 = 0.$

8. $(a^2 - x^2)y'^2 + 2xyy' + x^2 = 0.$

9. $x^3y'^2 + x^2yy' + a = 0.$

10. $4x^2y - 2x^3y' + ay'^2 = 0.$

IV·3. Equation solvable for *x*. A method, entirely analogous to that of the previous section, can be deduced in case the equation can be solved for x.* Suppose the equation to be in the form

$$x = \theta(y, y'). \tag{1}$$

Differentiating with respect to y, we have

$$\frac{dx}{dy} = \frac{1}{y'} = \frac{\partial\theta}{\partial y} + \frac{\partial\theta}{\partial y'}\frac{dy'}{dy}. \tag{2}$$

Here x no longer appears, and we may look upon this equation as one of the first order in y and y'. If we can integrate it, we obtain a relation involving an arbitrary constant,

$$\chi(y, y', c) = 0. \tag{3}$$

On eliminating y' between (1) and (3), we have the general solution.

Exercise **1.** $a^2yy'^2 - 2xy' + y = 0$. **2.** $x + yy'(2y'^2 + 3) = 0$.

3. $y^2y'^2 - 3xy' + y = 0$. **4.** $y'^3 - 4xyy' + 8y^2 = 0$.

5. $x^2y'^2 + (2xy - y^2)y' + y^2 = 0$.

6. $2yy'^3 - 3xy' + 2y = 0$. **7.** $y^2y'^3 + 2xy' - y = 0$.

8. Find the family of curves for which the length of the normal (from the curve to the axis of x) is equal to the square root of the length of its intercept on the axis of x.

IV·4. Clairaut equation.† If the equation is of the form

$$y = xy' + f(y'), \tag{1}$$

where $f(y')$ is a function of y' only, the solution is obtained so readily that special attention should be given to this form of equation, in order that it may be recognized at once.

---

* Here, also, it is not necessary to solve for x. The method applies whenever the equation is linear in x, with the coefficient of x a constant.

† This form of equation is named after Alexis Claude Clairaut (1713–1765). He was the first to apply the process of differentiation (§§IV·2, IV·3) to the solution of differential equations. His application of this method to the equation that bears his name appeared in *Histoire de l'Académie des Sciences de Paris*, 1734.

Using the method of §IV·2, we have

$$y' = y' + [x + f'(y')]\frac{dy'}{dx},$$

or

$$[x + f'(y')]\frac{dy'}{dx} = 0.$$

Neglecting, for the present, the factor $x + f'(y')$, which involves no differential expressions (see §V·3, Note), we have

$$\frac{dy'}{dx} = 0; \text{ whence } y' = c.$$

Putting this value in (1), we have

$$y = cx + f(c), \tag{2}$$

which is the general solution of (1).

The student should be able to recognize a Clairaut equation, even when it is not solved for y. Obviously, what characterizes an equation of this form is that x and y occur only in the combination $y - xy'$. Hence any function of $y - xy'$ and y' equated to zero, viz. $F(y - xy', y') = 0$, is a Clairaut equation, and its solution is $F(y - cx, c) = 0$.

***Exercise* 1.** $$(xy' - y)^2 = y'^2 + 1.$$

This is obviously in Clairaut's form. Hence its solution is known at once to be

$$(cx - y)^2 = c^2 + 1.$$

2. $xy'^2 - yy' + 3 = 0$. **3.** $(x^2 + 1)y'^2 - 2xyy' + y^2 = 1$.

4. $x^2y'^2 - 2(xy - 4)y' + y^2 = 0$.

A common practice, when endeavoring to solve an equation which does not come under a known head, is to introduce a new variable or variables. In the following set of exercises the suggested transformations reduce the equations to Clairaut's form. The student should understand that it is not possible, in general, to predict what form an equation will take when new variables are introduced. In certain cases, as in a number of those enumerated in Chapter II, the resulting form of the equation is known. But, no general rule can be given to determine when a differential equation of the first order can be reduced to a Clairaut equation. As a matter of fact, it is only in exceptional cases that such a transformation is possible. The purpose of the exercises below is twofold: first, to have the student obtain further practice in actually carrying out a transformation of variables, and, second, to have him recognize a Clairaut equation when one arises.

5. $4x^2y = 2x^3y' - ay'^2$. (Ex. 10, §IV·2.)

Let $x^2 = u$. Then $y' = \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = 2x\dfrac{dy}{du}$, and the equation reduces to

$$y = u\frac{dy}{du} - a\left(\frac{dy}{du}\right)^2.$$

The solution is

$$y = cu - ac^2, \quad \text{or} \quad y = cx^2 - ac^2.$$

6. $4e^{2y}y'^2 + 2xy' - 1 = 0$.

Let $e^{2y} = v$. Then $y' = \dfrac{dy}{dx} = \dfrac{1}{2v}\dfrac{dv}{dx}$, and the equation becomes

$$v = x\frac{dv}{dx} + \left(\frac{dv}{dx}\right)^2.$$

Its solution is

$$v = cx + c^2, \quad \text{or} \quad e^{2y} = cx + c^2.$$

7. $y = 2xy' + y^2y'^3$. (Ex. 7, §IV·3.) [Let $y^2 = v$.]

8. $a^2yy'^2 - 2xy' + y = 0$. (Ex. 1, §IV·3.) [Let $y^2 = v$.]

9. $xy^2y'^2 - y^3y' + a^2x = 0$. [Let $x^2 = u$, $y^2 = v$.]

10. $(x^2 + y^2)(1 + y')^2 - 2(x + y)(1 + y')(x + yy') + (x + yy')^2 = 0$.
[Let $x + y = u$, $x^2 + y^2 = v$.]

11. $(xy' - y)^2 = x^2(2xy - x^2y')$. [Let $y = vx$.]

12. $y^2y'^2 - 2xyy' + 2y^2 - x^2 = 0$. (Ex. 8, §IV·1.)
[Let $x^2 + y^2 = v$.]

13. $(a^2 - x^2)y'^2 + 2xyy' + x^2 = 0$. (Ex. 8, §IV·2.)
[Let $x^2 = u$, $2y = v$.]

14. $y + xy' = x^4y'^2$. (Ex. 5, §IV·2.) [Let $xy = v$.]

15. $4e^{2y}y'^2 + 2e^{2x}y' - e^{2x} = 0$. [Let $e^x = u$, $e^{2y} = v$.]

16. $e^{2y}y'^3 + (e^{2x} + e^{3x})y' - e^{3x} = 0$. [Let $e^x = u$, $e^y = v$.]

17. $x^3y'^2 + x^2yy' + a = 0$. (Ex. 9, §IV·2.) [Let $1/x = u$.]

18. $2x^3y - x^4y' + 3y'^3 = 0$. (Ex. 7, §IV·2.) [Let $x^2 = u$.]

19. $4x^2y^2y'^2 - 2(x^3 + 2y^2)xyy' + y^2(2x^3 + y^2) = 0$. [Let $y^2 = xv$.]

20. $2y^2y'^2 + 2xyy' + x^2 + y^2 - 1 = 0$. [Let $x^2 + y^2 - 1 = v$.]

IV·5. Summary. Three methods* for solving a differential equation of the first order and higher degree than the first have been developed in this chapter. In actual practice, these should be tried in the following order:

1° Solve for y', and then solve the resulting equations of the first degree (§IV·1).

2° Solve for y, differentiate with respect to x, integrate, and eliminate y' between this solution and the original equation (§IV·2).

3° Solve for x, differentiate with respect to y, integrate, and eliminate y' between this solution and the original equation (§IV·3).

Evidently methods 2° and 3° may also be used for differential equations of the first degree.

Clairaut's form (§IV·4) has been given special prominence in this chapter because of the ease of finding its solution. It is, of course, solved by method 2°.

If none of these methods is applicable, a substitution may be sought to bring the equation into manageable form.

There are certain cases when we can tell in advance that some or all of these methods apply. Here the difficulties are those of Algebra or the Integral Calculus, and not of the Differential Equations. For example, consider the following cases:

(*a*) If the equation in y' is algebraic and all the coefficients are homogeneous and of the same degree in x and y, then, on dividing by the leading coefficient, all the coefficients are homogeneous and of zero degree. Hence, if we can solve for y' (which is an algebraic process), we obtain y' as homogeneous functions of x and y of degree zero, and the resulting equations are homogeneous (§II·2).

---

* It is almost needless to remark that these methods are not mutually exclusive. Two, or all three, methods may be applicable to the same equation.

As noted before, the differential equation assumes the form $f(y', y/x) = 0$ when divided by any one of its coefficients. Hence, if we solve it for y (or y/x), we obtain $y = x\psi(y')$. Differentiating, we have

$$y' = \psi(y') + x\psi'(y')\frac{dy'}{dx},$$

where the variables are separable.

It will be left, as an exercise, to show in a similar manner that method 3° also may be employed.

Hence, in this case, all three general methods are applicable.

(*b*) If x is absent, the equation is of the form $f(y', y) = 0$. Solving for y', we have $y' = dy/dx = \phi(y)$. Here the variables are separable.

Again, solving for y, we have $y = \psi(y')$. Differentiating, we have $y' = \psi'(y')\,dy'/dx$. Here also the variables are separable.

Hence, in this case, methods 1° and 2° are applicable.

(*c*) If y is absent, the equation is of the form $f(y', x) = 0$. Let the student show, as an exercise, that in this case methods 1° and 3° are applicable.

(*d*) If the equation is of the first degree in x and y, thus,

$$xf_1(y') + yf_2(y') + f_3(y') = 0,\text{*}$$

it is readily seen that method 2° is applicable. For, solving for y, we have

$$y = x\psi_1(y') + \psi_2(y').$$

Differentiating, we have

$$y' = \psi_1(y') + [x\psi_1'(y') + \psi_2'(y')]\frac{dy'}{dx};$$

With y' as the independent variable, this may be written

$$\frac{dx}{dy'} + x\frac{\psi_1'(y')}{\psi_1(y') - y'} = \frac{\psi_2'(y')}{y' - \psi_1(y')},$$

which is linear (§II·9).

Exercise **1.** $y^2(1 + y'^2) = a^2$. **2.** $yy' = (x - b)y'^2 + a$.

3. $x^3y'^2 + x^2yy' + 1 = 0$. **4.** $3xy'^2 - 6yy' + x + 2y = 0$.

5. $y = y'^2(x + 1)$.

6. $xyy'^2 + (x^2 - y^2 - a^2)y' - xy = 0$. [Let $x^2 = u$, $y^2 = v$.]

7. $y'^2 + 2yy'\cot x = y^2$.

8. $(1 + x^2)y'^2 - 2xyy' + y^2 - 1 = 0$.

---

* Clairaut's equation is a special case of this.

9. $x^2y'^2 - 2(xy + 2y')y' + y^2 = 0.$

10. $yy'^2 + x^3y' - x^2y = 0.$ [Let $x^2 = u$, $y^2 = v$.]

11. $x^2y'^2 - 3xyy' + x^3 + 2y^2 = 0.$ [Let $y = xv$.]

12. $x^2y'^2 + (2xy - x^2)y' + y^2 = 0.$ [Let $xy = v$.]

13. $y^2y'^2 - 4xyy' + 2x^2 + y^2 = 0.$

14. $x^2y'^2 - 2xyy' + y^2 = x^2y^2 + x^4.$

15. $\dfrac{y - xy'}{\sqrt{1 + y'^2}} = f(\sqrt{x^2 + y^2}).$ [Let $x = \rho \cos \theta$, $y = \rho \sin \theta$.]

16. Find the equation of the curves for which the distance of the tangent from the origin varies as the distance of the point of contact from the origin.

17. Find the equation of the curves such that the square of the length of arc measured from a fixed point is a constant times the ordinate of the variable point. [Let the constant factor be $8k$.]

18. Find the equation of the curves down each of whose tangents a particle, starting from rest, will slide to the axis of x in the same time. [We saw in Ex. 2, §III·4, that the distance covered in the time t equals $\frac{1}{2}gt^2 \sin \alpha$. Here $\sin \alpha = \dfrac{y'}{\sqrt{1 + y'^2}}$, and the distance covered is $\dfrac{y\sqrt{1 + y'^2}}{y'}$, §III·2, (*g*).]

CHAPTER V

SINGULAR SOLUTIONS

V·1. Singular solution. Let

$$\phi(x, y, c) = 0 \tag{1}$$

be the general solution of

$$f(x, y, y') = 0. \tag{2}$$

If the family of curves defined by (1) *has an envelope, the equation of the latter is a solution of* (2).

It will be recalled that the *envelope* of a family of ∞^1 curves is defined as the curve (or curves) such that, at each of its points, it is tangent to a curve of the family.* Now, we have already noted (§III·1) that the slope of an integral curve of (2) at each of its points (x, y) is a value of y' determined by (1) for that pair of values of x and y. Since the slope of the envelope at any point is the same as that of the integral curve touching it there, the equation of the envelope must also satisfy the differential equation. Moreover, since the envelope is usually not a curve of the family [i.e., its equation cannot be obtained from (1) by assigning a definite value to the constant c], the equation of the envelope is usually not a particular solution. It is known as the *singular solution.*

We recall further that the envelope of the family of curves $\phi(x, y, c) = 0$ is obtained by eliminating c between

$$\phi(x, y, c) = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial c} = 0.$$

***Exercise* 1.** $xy'^2 - yy' + 1 = 0.$

Writing this in the form

$$y'(xy' - y) + 1 = 0,$$

---

* See, for example, the author's Calculus, §115.

we recognize it as a Clairaut equation, whose solution is

$$c^2x - cy + 1 = 0. \tag{3}$$

Differentiating (3) with respect to c, we have

$$2cx - y = 0. \tag{4}$$

Eliminating c between (3) and (4), we obtain

$$y^2 = 4x,$$

which is the singular solution. It will be left as an exercise to verify that this is a solution.

Exercise **2.** $xy'^2 - 2yy' + x = 0.$

This is the equation of Ex. 2, §IV·1. We saw there that its solution is

$$c^2x^2 - 2cy + 1 = 0.$$

Differentiating with respect to c, we have

$$cx^2 - y = 0.$$

Eliminating c, we have the singular solution

$$x^2 - y^2 = 0.$$

Verify that this is a solution of the given differential equation.

V·2. Discriminant. If $f(z)$ is a polynomial of the nth degree,

$$c_0z^n + c_1z^{n-1} + c_2z^{n-2} + \cdots + c_{n-1}z + c_n,$$

we have, by Taylor's theorem,

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \cdots + \frac{f^{(n)}(a)}{n!}h^n,$$

where $f'(a) = \left[\dfrac{df(z)}{dz}\right]_{z=a} = nc_0a^{n-1} + (n-1)c_1a^{n-2} + \cdots + 2c_{n-2}a + c_{n-1},$

$$f''(a) = \left[\frac{d^2f(z)}{dz^2}\right]_{z=a} = n(n-1)c_0a^{n-2} + (n-1)(n-2)c_1a^{n-3} + \cdots + 2c_{n-2},$$

. . . .

$$f^{(n)}(a) = \left[\frac{d^nf(z)}{dz^n}\right]_{z=a} = n!\,c_0.$$

Putting $h = z - a$, we have

$$f(z) = f(a) + f'(a)(z - a) + \frac{f''(a)}{2!}(z - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z - a)^n.$$

From this it is obvious that, if a is a root of $f(z) = 0$, i.e., if $f(a) = 0$, $f(z)$ contains $z - a$ as a factor; and conversely, in order that $f(z)$ should contain the factor $z - a$, we must have $f(a) = 0$. Similarly, if a is a double root, $(z - a)^2$ is a factor of $f(z)$, so that we must have $f(a) = 0$ and $f'(a) = 0$; conversely, if $f(a) = 0$ and $f'(a) = 0$, $f(z)$ contains $(z - a)^2$ as a factor, and a is a double root.* The necessary and sufficient condition, then, that $f(z) = 0$ have a repeated root, a, is that $f(z)$ and $f'(z)$ have the corresponding factor $z - a$ in common. This condition obviously depends upon the coefficients of $f(z)$. *That rational, entire*† *function of the coefficients whose vanishing expresses the necessary and sufficient condition that* $f(z) = 0$ *shall have a repeated root* is called the *discriminant* of $f(z)$. It can be shown to be equal to the product of the squares of the differences of the various roots of the equation (multiplied by a power of c_0 to avoid fractions). It can be calculated in various ways. The process of finding the greatest common divisor will show when $f(z)$ and $f'(z)$ have a common factor $z - a$. But this process is apt to introduce extraneous factors. A better way is to eliminate z between $f(z) = 0$ and $f'(z) = 0$ [or between $nf(z) - zf'(z) = 0$ and $f'(z) = 0$, which are both of degree $n - 1$]. The result of this elimination is a relation among the coefficients of $f(z)$, which expresses the condition that $f(z) = 0$ and $f'(z) = 0$ can be satisfied simultaneously. If all the terms of this relation are brought over to one side of the equation and the expression is cleared of fractions and radicals, we have the discriminant equated to zero. We shall call this the *discriminant relation.* Various methods for eliminating the variable from two polynomials involving it are given in the Theory of Equations. It will be sufficient to recall here that the discriminant of the quadratic $az^2 + bz + c$ is

---

* If $f''(a) = 0$ also, $f(z)$ contains $(z - a)^3$ as a factor, and a is a triple root. Hence, $f(a) = 0$, $f'(a) = 0$, but $f''(a) \neq 0$ are the conditions that a be exactly a double root and not one of higher order.

In an entirely analogous manner it can be seen that $f(a) = 0$, $f'(a) = 0$, $f''(a) = 0, \cdots, f^{(r-1)}(a) = 0$, but $f^{(r)}(a) \neq 0$ are the conditions that a be an r-fold root.

† The *rational* operations of Algebra are addition, subtraction, multiplication, and division. The first three of these are said to be *entire*; they do not introduce fractions.

$$4ac - b^2,$$

while that of the cubic $az^3 + bz^2 + cz + d$ is

$$27a^2d^2 + 4ac^3 + 4b^3d - 18abcd - b^2c^2.$$

In the case of $\phi(x, y, c) = 0$, looked upon as an equation in c, the coefficients are functions of x and y. It may be possible to find values of x and y, such that this equation shall have equal roots. Looked upon geometrically, this means that there may be points through which there pass a smaller number of integral curves than usual; for to each value of c there corresponds an integral curve. The discriminant relation, in this case, is the locus of such points.

V·3. Singular solution obtained directly from the differential equation. Since the problem of finding the equation of the envelope of $\phi(x, y, c) = 0$ is identical with that of finding the discriminant relation, we see that through each point of the envelope there pass a smaller number of integral curves than through points of general position in the plane; that is, at least two of the integral curves through each point of the envelope coincide.

Hence, at least two of the directions, determined by the differential equation, must coincide at such a point; that is, the differential equation $f(x, y, y') = 0$, looked upon as an equation in y', must have a repeated root for values of x and y satisfying the equation of the envelope. As a consequence, the result of eliminating y' between $f(x, y, y') = 0$ and $\partial f/\partial y' = 0$ must also give the equation of the envelope and, therefore, the singular solution, whenever there is one.

As an example, consider the family of circles $(x - c)^2 + y^2 = r^2$, where r is a fixed constant and c is the parameter. The c-discriminant relation is $y^2 = r^2$, or $y = \pm r$. Each of these lines is tangent to every circle of the family. Through each point on the envelope there passes one and only one circle of the family, while through any other point in the plane there pass two circles (either real or imaginary, depending upon whether the absolute value of the ordinate of the point is less than r or greater than r).

The differential equation of this family of circles is readily found to be

$$y^2(1 + y'^2) = r^2.$$

In general, this equation determines two distinct directions at a point (x, y). But, if $y = \pm r$, this equation determines the unique value $y' = 0$. Also, when $y = 0$, the two values $\pm \infty$ for y' determine the same direction. For this exceptional case see §V·4.

NOTE. In the previous chapter we came upon certain factors in the course of solving equations, which, while they would have led to solutions, did not contain arbitrary constants, and were therefore neglected at that time. It will now be understood that these factors usually lead to singular solutions. Thus, in the case of an equation in Clairaut's form (§IV·4),

$$y = xy' + f(y'), \tag{1}$$

the neglected factor is

$$x + f'(y') = 0. \tag{2}$$

But, this is exactly the derivative of (1) with respect to y'. So that, if we eliminate y' between (1) and (2), we obtain the singular solution. Again, in Ex. 1 (§IV·2), we neglected the factor $4y' + 1 = 0$. Eliminating y' between this and the original equation, we have $x + 4y = 0$, which is a singular solution of the equation.

Remark. From the fact that two roots of an equation can be equal only when there are as many as two roots, no singular solution can exist in the case of equations of the first order and degree. But it may, and not infrequently does, happen that equations of a higher degree than the first have no singular solutions.

It may be remarked further that at times the singular solution is more interesting than the general solution.

For example, let us ask for that curve which has the property of having the length of each of its tangents, intercepted by the coördinate axes, equal to the same constant l.

From formula (*i*), §III·2, we have

$$(y - xy')^2(1 + y'^2) = l^2y'^2.$$

Since this is in Clairaut's form (§IV·4), the general solution, when solved for y, is seen at once to be

$$y = cx \pm \frac{cl}{\sqrt{c^2 + 1}}.$$

This represents a family of lines whose length intercepted by the axes is l. The equation of the curve, in which we are especially interested, is given by the singular solution. This is obtained by finding either the y'- or the c-discriminant relation (the two being identical in the case of an equation in Clairaut's form). Each one is

$$x^{2/3} + y^{2/3} = l^{2/3},$$

which is a hypocycloid of four cusps.

This may be shown as follows:
Starting with

$$(y - cx)^2(1 + c^2) = l^2c^2, \tag{3}$$

we have, upon differentiating with respect to c,

$$c(y - cx)^2 - x(y - cx)(1 + c^2) = l^2c. \tag{4}$$

Dividing (4) by (3), we have

$$\frac{c}{1 + c^2} - \frac{x}{y - cx} = \frac{1}{c}, \quad \text{or} \quad \frac{c^2}{1 + c^2} = \frac{y}{y - cx}, \quad \text{or} \quad \frac{1 + c^2}{c^2} = \frac{y - cx}{y};$$

whence

$$\frac{1}{c^2} = -\frac{cx}{y}, \quad \text{or} \quad c = -(y/x)^{1/3}.$$

Substituting this in (3), we have

$$(y + x^{2/3}y^{1/3})^2(1 + y^{2/3}/x^{2/3}) = l^2y^{2/3}/x^{2/3}.$$

Multiplying both sides by $x^{2/3}/y^{2/3}$, we have

$$(y^{2/3} + x^{2/3})^3 = l^2, \quad \text{or} \quad x^{2/3} + y^{2/3} = l^{2/3}.$$

Exercise **1.** Find the curve for which the product of the perpendiculars drawn from two fixed points to any tangent is the same for all tangents of the curve.

2. Find the curve all of whose tangents are equidistant from the origin.

3. Find the curve for which the area enclosed between any tangent and the coördinate axes is a^2, the same number for all tangents of the curve.

4. Find the curve such that the sum of the intercepts of a tangent on the coördinate axes is the same for all tangents of the curve.

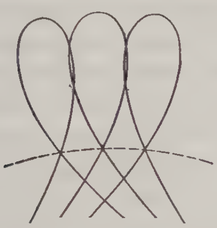
5. Integrate the following equations and examine them for singular solutions:

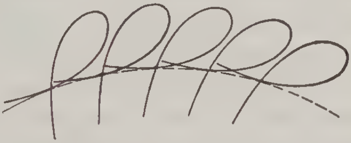
(*a*) $x^2y'^2 - 2(xy - 2)y' + y^2 = 0$, (*b*) $xy'^2 - y = 0$,
(*c*) $xy'^2 - yy' + 1 = 0$, (*d*) $(y - xy')^2 = b^2 + a^2y'^2$,
(*e*) $yy'^2 - 2xy' + y = 0$, (*f*) $(1 - x^2)y'^2 + y^2 - 1 = 0$.

V·4. Extraneous loci. We have noted that the *c*-discriminant relation is the equation of the locus of points through which pass a smaller number of integral curves than usual. We have also seen that the y'-discriminant relation is the locus of points at which a smaller number of directions than usual is determined by the differential equation. Points on the envelope of the family of integral curves lie on both of these loci. The question arises, do either of these loci contain points which are not on the envelope?

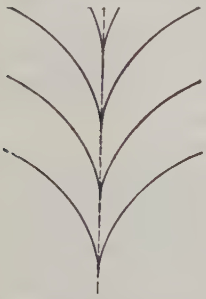
If an integral curve has a double point, at this point there will be two branches of the curve with distinct tangents. Since there are only n values of y' (if the differential equation is of the

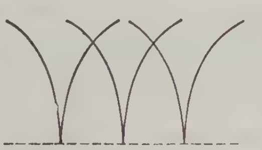
nth degree), there can be only $n - 2$ other curves through this point. Hence this point must be on the locus of the c-discriminant relation. And if there are ∞^1 integral curves having double points, or nodes as they are sometimes called, the locus of these points (known as the *nodal locus*) will be given by the c-discriminant relation. Excepting in the unusual case where this locus is also an envelope, its equation will not satisfy the differential equation. The two cases are illustrated by the accompanying figures. Moreover, since, in the general case, there are still n distinct values of y' on the nodal locus, the equation of the latter does not appear in the y'-discriminant relation.





A special case of a double point is a cusp, which may be looked upon as the limiting case of a double point, where the loop has shrunk up to the point and the two branches of the curve have become tangent. The equation of the locus of the cusps of the integral curves, known as the *cuspidal locus*, found when the equation of the envelope is sought, will be a solution only in case this locus is also an envelope. Otherwise, it is not a solution. Note the accompanying figures.





In the case of a cusp, not only is the number of integral curves through that point at least one less than the usual number, that is, not only does the c-discriminant vanish at this point, but two values of y' are equal there, since the tangents to the two branches of the curve coincide, that is, at such a point the y'-discriminant also vanishes. Hence the equation of the cuspidal locus must also appear in the y'-discriminant relation.

These extraneous loci, whose equations, in general, are not solutions, have arisen as results of peculiarities of the integral curves. Thus, if the integral curves are known to have no double points or cusps, it is clear there can be no nodal or cuspidal loci. But, an extraneous locus may arise, irrespective of the character of the integral curves. Wherever two distinct integral curves are tangent to each other, while the number of curves through that point is unaffected, the number of distinct values of y' is diminished. Hence the y'-discriminant vanishes at that point, and the locus of such points, if it exists, will be given by the y'-discriminant relation. This locus is known as the *tac-locus,* and its equation does not, in general, satisfy the differential equation; neither does it appear in the c-discriminant relation.

Thus, in the case of the family of circles referred to in §V·3, the y'-discriminant relation is found to be $y^2(y^2 - r^2) = 0$; here $y = \pm r$ is the envelope, while $y = 0$ is the tac-locus. By actual trial, it is found that $y = 0$ does not satisfy the differential equation.

It may be noted further that each of the factors $y + r$ and $y - r$ appears once in both the c- and y'-discriminants, while the factor y appears squared in the y'-discriminant.

These facts are consistent with the general principle proved by Professor M. J. M. Hill in the *Proceedings of the London Mathematical Society,* Vol. 19 (1888), p. 561. In this memoir he showed that, in general, the factors in the c-discriminant corresponding to the envelope, nodal, and cuspidal loci occur once, twice, and three times respectively; while those corresponding to the envelope, tac-locus, and cuspidal locus occur once, twice, and once respectively, in the y'-discriminant.

At times, as the parameter approaches a certain value, the curves of the family approach a limiting one, usually different in shape from all the others. Occasionally, this particular curve of the family has the property of meeting all the others in the same points.* Excepting at these points (through which there is an infinite number of curves), a smaller number of curves than usual pass through every point of the particular curve. Hence the equation of the latter is given by both the y'- and c-discriminant relations. Moreover, it is found that the factor corresponding to this particular solution appears once in the c-discriminant and three times in the y'-discriminant.

Thus, in the case of the differential equation

$$x^3y'^2 + x^2yy' + a = 0 \quad \text{(Ex. 9, §IV·2)}$$

the y'-discriminant is

$$x^3(xy^2 - 4a).$$

The integral curves are the equilateral hyperbolas

$$ax + cxy + c^2 = 0,$$

each of which has the axis of y as an asymptote. This is also a particular curve, obtained by putting $c = 0$. This particular curve is thus tangent to each of the others at infinity. No other integral curve passes through any point on $x = 0$ at a finite distance from the origin. The c-discriminant is

$$x(xy^2 - 4a).$$

Here $xy^2 - 4a = 0$ is the singular solution.

The following table indicates the number of times the factors corresponding to these various loci usually† appear in the c- and y'-discriminants:

---

* See H. Bateman, *Differential Equations* (1918), §§38, 39. Also, M. Petrovitch, *Mathematische Annalen*, Vol. 50 (1918), p. 103.

† Instances, where these numbers do not seem to hold, arise occasionally. For example, see H. T. H. Piaggio, *Differential Equations*, 2nd Ed., §161.

c-discriminant		y'-discriminant	
envelope	1	envelope	1
particular curve	1	particular curve	3
nodal locus	2	tac-locus	2
cuspidal locus	3	cuspidal locus	1

In case any locus comes under two headings, the corresponding factor occurs the number of times it should for each of them; thus, if a tac-locus is also an envelope, that factor occurs once in the c-discriminant and three times in the y'-discriminant; again, if the cuspidal locus is an envelope, the corresponding factor occurs four times in the c-discriminant and twice in the y'-discriminant; and so on.

If the same integral curve is determined by two distinct values of the arbitrary constant, the equation of this curve will appear as a tac-locus; that is, its left-hand member will appear twice as a factor in the y'-discriminant.

Thus, consider the family of nodal cubics

$$y^2 = 2x(x - c)^2.$$

Its differential equation is

$$(2xy' - y)^2 = 8x^3.$$

The c- and y'-discriminants are, neglecting constant factors,

$$xy^2 \quad \text{and} \quad x^5$$

respectively.

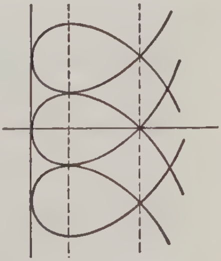
Here the factor y, appearing twice in the c-discriminant and no times in the y'-discriminant, suggests that $y = 0$ is a nodal locus. Examination of the family of integral curves verifies this fact.

Further examination of this family shows that the axis of y is a particular curve, in the sense that it is tangent to each of the other integral curves at the origin. This will account for the factor x occurring once in the c-discriminant and three times in the y'-discriminant. But, in addition, $x = 0$ may be obtained from the general solution by putting $c = +\infty$ and $c = -\infty$. Hence, $x = 0$ plays the rôle of a tac-locus. This will account for the factor x occurring two additional times in the y'-discriminant, or five times in all.

Attention should be called to the fact that the number of times a factor appears in each of the discriminants is often helpful in enabling one to classify the loci obtained by equating them to zero. But, since the same combinations of numbers may often be interpreted in several ways, it is obvious that consideration of these combinations is to be used *as a guide only*. A drawing of a representative set of integral curves will usually aid materially in interpreting the analytic result.

Moreover, the process of finding discriminants is frequently beset with chances to introduce or to drop a factor, so that, unless great care is taken, the number of times a factor is found to occur may not be the correct one, and inferences drawn from it will be false. In actual practice, it is desirable to find both the y'- and the c-discriminants, and then to test their various factors, equated to zero, to see if they satisfy the equation. In this way, if a factor has been lost in either one of the discriminants, its appearance in the other will keep it from being lost as a solution.

In certain cases there can be no doubt. Thus, if the degree of the equation is two or three, the use of the formulas mentioned in §V·2 will give all the factors occurring the correct number of times. Again, in case the integral curves are straight lines or conics, there can be no nodal or cuspidal loci.



Exercise 1. $xy'^2 - (x-1)^2 = 0$.

The general solution is readily seen to be

$$9(y+c)^2 = 4x(x-3)^2,$$

which is the equation of a family of nodal cubics, each of which is tangent to the axis of y and has its node at the point $(3, c)$. See figure, page 97.

The y'-discriminant relation is $x(x-1)^2 = 0$, while the c-discriminant relation is $x(x-3)^2 = 0$.

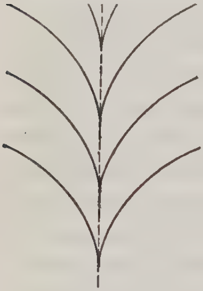
Here $x = 0$ is common to the two. It also satisfies the equation. (For the line $x = 0$, $y' = \infty$ at every point.) Hence it is the singular solution.

But $x - 1 = 0$ occurs in the y'-discriminant only. It is the tac-locus. (Notice that this factor occurs twice.)

And $x - 3 = 0$ occurs in the c-discriminant only. It is the nodal locus. (Notice that this factor occurs twice.)

***Exercise* 2.** $27xy'^3 = 8$.

The general solution is easily found to be



$$(y + c)^3 = x^2,$$

a family of semi-cubical parabolas. If we neglect constant factors, the c- and y'-discriminants may be written,

$$x^4 \quad \text{and} \quad x^2$$

respectively. Examination of the family of integral curves shows that $x = 0$ is the equation of the envelope. It satisfies the differential equation, since $y' = \infty$ along this line is consistent with the differential equation.

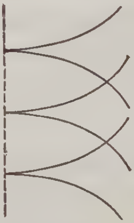
Moreover, $x = 0$ is the equation of the cuspidal locus. See figure.

***Exercise* 3.** $4y'^2 = 9x$.

The general solution is

$$(y + c)^2 = x^3.$$

Here the y'-discriminant relation is $x = 0$, and the c-discriminant relation is $x^3 = 0$.



It is obvious that $x = 0$ does not satisfy the equation. It is a cuspidal locus. See figure.

4. $yy'^2 = 1$.

5. $y(3 - 4y)^2y'^2 = 4(1 - y)$.

6. $(a - x)y'^2 + (y - x)y' + y = 0$.

7. Examine the following equations for singular solutions and extraneous loci:

§IV·1, Ex. **3**, **8**.
§IV·2, Ex. **1**, **4**, **5**, **6**, **7**, **8**, **10**.
§IV·3, Ex. **1**, **3**, **4**, **5**, **6**, **7**, **8**.
§IV·4, Ex. **1**, **2**, **3**, **4**, **9**, **10**, **11**, **19**, **20**.
§IV·5, Ex. **2**, **3**, **4**, **5**, **8**, **12**, **13**.

V·5. Summary. We have seen that the equation of the singular solution (or of the envelope) is given by both the c- and y'-discriminant relations (§§V·1, V·3). Moreover, the c-discriminant relation gives rise to the nodal and cuspidal loci, while the y'-discriminant relation gives rise to the cuspidal and tac-loci, and both of them, at times, give rise to a particular solution (§V·4). For the number of times the corresponding factors occur in each discriminant, see table in §V·4.

The theory as given here was first developed by Arthur Cayley (1821–1895), *Messenger of Mathematics*, Vol. II (1873), p. 6; Vol. VI, p. 23. For illustrative examples, see J. W. L. Glaisher, *Messenger of Mathematics*, Vol. XII, p. 1.

It should be noted that, in general, a differential equation has no singular solution. For, solving $f(x, y, y') = 0$ and $\partial f/\partial y' = 0$ for y and y', we have

$$y = \phi(x), y' = \phi_1(x).$$

In order that this value of y be a solution, we must have

$$\phi_1(x) = \frac{d\phi(x)}{dx},$$

which is not true, in general. Darboux proved that, in general, the result of eliminating y' from the above two equations is the equation of the cuspidal locus. (*Bulletin des Sciences Mathématiques*, Vol. IV, 1873, p. 158.) Picard also gives a proof of this in his *Traité d'Analyse*, Vol. III, Chap. III. See also, for example, Fine's article in the *American Journal of Mathematics*, Vol. XII; Chrystal, *Nature*, 1896; Liebmann, *Differentialgleichungen*, p. 95. For singular solutions of differential equations involving the derivative transcendentally, see Professor M. J. M. Hill, *Proceedings of the London Mathematical Society*, Series 2, Vol. 17 (1918), p. 149. This memoir contains an account of the development of the theory of singular solutions. In this connection, reference may also be made to S. Rothenberg, "Geschichtliche Darstellung der Entwicklung der Theorie der singulären Lösungen," *Abhandlungen zur Geschichte der Mathematik, Leipzig*, Vol. 20 (1908), p. 3.

CHAPTER VI

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

VI·1. General linear differential equation. A *linear differential equation* is one which is of the first degree in the dependent variable and all of its derivatives. Its general type is

$$(1)\quad X_0y^{(n)} + X_1y^{(n-1)} + X_2y^{(n-2)} + \cdots + X_{n-1}y' + X_ny = X,$$

where $X_0, X_1, X_2, \cdots, X_n, X$ are functions of x or constants. If we write $y' = Dy$, $y'' = D^2y, \cdots, y^{(n)} = D^ny$, we can write (1) in the following convenient form,

$$(X_0D^n + X_1D^{n-1} + X_2D^{n-2} + \cdots + X_{n-1}D + X_n)y = X,$$

or

$$F(D)y = X,$$

where $F(D)$ is the polynomial $X_0D^n + X_1D^{n-1} + \cdots + X_n$ which represents symbolically the differential operator

$$X_0\frac{d^n}{dx^n} + X_1\frac{d^{n-1}}{dx^{n-1}} + \cdots + X_{n-1}\frac{d}{dx} + X_n.$$

Two properties of linear differential equations which are of service in their solution deserve especial mention here:

1° Suppose $X = 0$. In this case the equation is said to be a *homogeneous* linear differential equation, since all of its terms are of the first degree in y and its derivatives. (When not homogeneous, the equation is said to be a *complete* linear differential equation.) If $y = y_1$ satisfies the equation, $y = c_1y_1$, where c_1 is a constant, will also satisfy it. For, since $D^k(c_1y_1) = c_1D^ky_1$, $F(D)(c_1y_1) = c_1F(D)y_1$. But, by hypothesis, $F(D)y_1 = 0$; hence $F(D)(c_1y_1) = 0$.

Moreover, if $y = y_2$ is also a solution, and c_1 and c_2 are any constants, $y = c_1y_1 + c_2y_2$ will be a solution. For, since the derivative of a sum is equal to the sum of the derivatives, i.e., $D^k(y_1 + y_2) = D^ky_1 + D^ky_2$, we have

$$\begin{aligned} F(D)(c_1y_1 + c_2y_2) &= F(D)(c_1y_1) + F(D)(c_2y_2) \\ &= c_1F(D)y_1 + c_2F(D)y_2 = 0. \end{aligned}$$

Similarly, if we know r particular integrals,[*] $y_1, y_2, \cdots, y_r$,

$$y = c_1y_1 + c_2y_2 + \cdots + c_ry_r$$

will be a solution. Since the general solution of a differential equation of the nth order is a solution which involves n independent arbitrary constants, we have the property:

A. If $y_1, y_2, \cdots, y_n$ are n linearly independent[†] *particular integrals of a homogeneous linear differential equation of the nth order, the function $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is its general integral.*

If the particular integrals are not linearly independent, the solution found above will not be the general solution. Thus, suppose there exists the relation $a_1y_1 + a_2y_2 + \cdots + a_ny_n \equiv 0$, where the a's are constants, and not all of them are zero. If a_n is different from zero,

$$y_n \equiv -\frac{a_1}{a_n}y_1 - \frac{a_2}{a_n}y_2 - \cdots - \frac{a_{n-1}}{a_n}y_{n-1},$$

and the integral becomes

$$\left(c_1 - \frac{a_1}{a_n}c_n\right)y_1 + \left(c_2 - \frac{a_2}{a_n}c_n\right)y_2 + \cdots + \left(c_{n-1} - \frac{a_{n-1}}{a_n}c_n\right)y_{n-1},$$

where only $n - 1$ independent constants are involved.

---

[*] If $y = y_k$ is a solution of a linear differential equation, we shall say that y_k is an *integral* of the equation.

[†] The n functions $y_1, y_2, \cdots, y_n$ of a single variable are said to be *linearly independent* if it is impossible to find n constants $a_1, a_2, \cdots, a_n$ such that $a_1y_1 + a_2y_2 + \cdots + a_ny_n$ shall vanish for all values of the variables. Thus, $y_1 = 2x - x^2$, $y_2 = x + x^2$, $y_3 = x$ are evidently not linearly independent, since $y_1 + y_2 - 3y_3 \equiv 0$; i.e., it equals zero for all values of x, or, as it is usually expressed, $y_1 + y_2 - 3y_3$ vanishes identically.

For a formal statement and a proof of the condition for linear independence of a set of functions of a single variable, see Note II in the Appendix.

Remark. Attention should be called to the fact that it is immaterial how these particular integrals are obtained. We shall see that in a class of equations of frequent occurrence, these can be found by purely algebraic means; in other cases, some of them can be found by inspection.

For convenience of language, the general integral of the homogeneous linear differential equation, obtained from (1) by replacing the right-hand member by zero, is spoken of as the *complementary function* of (1).

2° If $Y \equiv c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the complementary function of (1), and if we know (no matter by what means) a particular integral, V, then $Y + V$ is the general integral of (1). For, since the equation is linear,

$$f(D)(Y + V) = f(D)Y + f(D)V = 0 + X = X.$$

Hence the property:

B. The general integral of a complete linear differential equation is equal to the sum of its complementary function and any particular integral.

VI·2. Linear differential equation with constant coefficients.* Complementary function.

Consider the equation

$$k_0y^{(n)} + k_1y^{(n-1)} + k_2y^{(n-2)} + \cdots + k_{n-1}y' + k_ny = X, \tag{1}$$

or

$$(k_0D^n + k_1D^{n-1} + k_2D^{n-2} + \cdots + k_{n-1}D + k_n)y = X,$$

or

$$f(D)y = X,$$

where $k_0, k_1, \cdots, k_n$ are constants.

We shall suppose, throughout §§VI·2, VI·3, VI·4, that $X = 0$. Then

$$f(D)y = 0. \tag{2}$$

---

* The methods given here are due to Leonhard Euler (1707–1783) and Jean-le-Rond d'Alembert (1717–1783).

Putting $y = e^{mx}$, we have $Dy = me^{mx}, \cdots, D^r y = m^r e^{mx}$; hence

$$f(D)e^{mx} = e^{mx} f(m).$$

For e^{mx} to be an integral of (2), m must satisfy the equation

$$(3) \quad f(m) \equiv k_0 m^n + k_1 m^{n-1} + k_2 m^{n-2} + \cdots + k_{n-1} m + k_n = 0.$$

Each root of (3) leads to an integral of (2). If the roots $m_1, m_2, \cdots, m_n$ are distinct, the integrals $e^{m_1 x}, e^{m_2 x}, \cdots, e^{m_n x}$ will be linearly independent, and, in consequence of A, §VI·1, $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$ will be the general integral of (2).

Remark. Equation (3), which is so readily obtained from equation (2), is often referred to as the *auxiliary equation.*[*]

***Exercise* 1.** $y'' - 3y' + 2y = 0.$

The auxiliary equation is $m^2 - 3m + 2 = 0$. Its roots are 1, 2. Hence the general solution is

$$y = c_1 e^x + c_2 e^{2x}.$$

***Exercise* 2.** $y'' - 6y' + 25y = 0.$

The roots of $m^2 - 6m + 25 = 0$ are $3 \pm 4i$, where $i = \sqrt{-1}$.

$$\therefore y = c_1 e^{(3+4i)x} + c_2 e^{(3-4i)x}, \quad \text{or} \quad y = e^{3x}(c_1 e^{4ix} + c_2 e^{-4ix}).$$

3. $y''' - y' = 0.$ **4.** $(D^3 - 2D^2 - D + 2)y = 0.$

5. $y'' + 4y = 0.$ **6.** $y''' + y'' - 4y' - 4y = 0.$

7. $(D^4 - 1)y = 0.$ **8.** $(6D^2 + D - 1)y = 0.$

VI·3. Roots of auxiliary equation repeated. If any roots of the auxiliary equation are repeated, the method of §VI·2 does not give us n linearly independent integrals, and, consequently, it does not give us the general solution. In this case we try the more general substitution $y = e^{mx}\phi(x)$, where $\phi(x)$ is a function of x entirely at our disposal. Then

$$\begin{aligned} Dy &= e^{mx}[m\phi + D\phi], \\ D^2 y &= e^{mx}[m^2\phi + 2mD\phi + D^2\phi], \\ D^3 y &= e^{mx}[m^3\phi + 3m^2 D\phi + 3mD^2\phi + D^3\phi], \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

---

[*] It is also known as the *characteristic equation.*

$$D^n y = e^{mx}\Big[m^n\phi + nm^{n-1}D\phi + \frac{n(n-1)}{2!}m^{n-2}D^2\phi + \cdots$$
$$+ \frac{n(n-1)\cdots(n-r+1)}{r!}m^{n-r}D^r\phi + \cdots + D^n\phi\Big].$$

Hence,

$$f(D)y = e^{mx}\Big[f(m)\phi + f'(m)D\phi + \frac{1}{2!}f''(m)D^2\phi + \cdots$$
$$+ \frac{1}{r!}f^{(r)}(m)D^r\phi + \cdots + \frac{1}{n!}f^{(n)}(m)D^n\phi\Big],$$

where $$f'(m) = \frac{d}{dm}f(m), \cdots, f^{(r)}(m) = \frac{d^r}{dm^r}f(m).$$

If m_1 is an r-fold root of $f(m) = 0$, we have (§V·2)

$$f(m_1) = 0, f'(m_1) = 0, \cdots, f^{(r-1)}(m_1) = 0, \text{ but } f^{(r)}(m_1) \neq 0.$$

In this case $f(D)y$ will vanish if $y = e^{m_1x}\phi(x)$, provided $D^r\phi \equiv 0$, whence all the higher derivatives of ϕ are also zero; i.e., provided $\phi = c_1x^{r-1} + c_2x^{r-2} + \cdots + c_{r-1}x + c_r$, where $c_1, c_2, \cdots, c_r$ are any constants whatever. Hence we see that, *if* m_1 *is an r-fold root of the auxiliary equation, not only is* e^{m_1x} *an integral of the equation, but so also are* xe^{m_1x}, $x^2e^{m_1x}$, $\cdots$, $x^{r-1}e^{m_1x}$, i.e., corresponding to an r-fold root we have r linearly independent integrals. Consequently, whether the roots of the auxiliary equation are repeated or not, the n linearly independent integrals necessary for obtaining the complementary function (A, §VI·1) are always supplied by the auxiliary equation.

***Exercise* 1.** $(4D^3 - 3D + 1)y = 0.$

The roots of $4m^3 - 3m + 1 = 0$ are $\frac{1}{2}, \frac{1}{2}, -1$. Hence the general solution is $y = (c_1 + c_2x)e^{x/2} + c_3e^{-x}$.

2. $(D^3 - D^2 - D + 1)y = 0.$ **3.** $(D^4 + 2D^3 - 2D - 1)y = 0.$

4. $(D^3 - 6D^2 + 9D)y = 0.$ **5.** $(D^4 + 3D^3 + 3D^2 + D)y = 0.$

6. $(D^4 - D^2)y = 0.$

VI·4. Roots of the auxiliary equation complex. If the coefficients of the differential equation are real, while some or all of the roots of the auxiliary equation are not, we can, by a proper arrangement of the terms in the complementary function, have the latter involve real terms only. For, if the auxiliary equation has a root $\alpha + i\beta$, it will also have $\alpha - i\beta$ as a root, since its coefficients are real. Two terms of the complementary function will then be

$$c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}, \quad \text{or} \quad e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}).$$

Now, $e^{i\beta x} = \cos \beta x + i \sin \beta x$, and $e^{-i\beta x} = \cos \beta x - i \sin \beta x$. Hence our pair of terms may be written

$$e^{\alpha x}[(c_1 + c_2) \cos \beta x + i(c_1 - c_2) \sin \beta x].$$

Putting

$$c_1 + c_2 = A \quad \text{and} \quad i(c_1 - c_2) = B,$$

we have

$$e^{\alpha x}(A \cos \beta x + B \sin \beta x),$$

where A and B are the two arbitrary constants.

Another form in which this may be written is

$$ae^{\alpha x} \sin (\beta x + \phi),^* \quad \text{or} \quad ae^{\alpha x} \cos (\beta x + \psi),$$

where a, ϕ, and ψ are arbitrary constants. For interpreting the solutions in physical problems, the latter forms are sometimes preferable.

It is not difficult to show that, in case a pair of such roots is repeated, the corresponding part of the complementary function may be put in the form

$$e^{\alpha x}(A_1 \cos \beta x + B_1 \sin \beta x) + xe^{\alpha x}(A_2 \cos \beta x + B_2 \sin \beta x),$$

---

* For, $A \cos \beta x + B \sin \beta x$ may be written

$$\sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}} \cos \beta x + \frac{B}{\sqrt{A^2 + B^2}} \sin \beta x\right).$$

Since the sum of the squares of $A/\sqrt{A^2 + B^2}$ and $B/\sqrt{A^2 + B^2}$ equals unity, these may be taken as the sine and cosine of some angle, e.g. ϕ. If we put $\sqrt{A^2 + B^2} = a$, our expression becomes $a(\sin \phi \cos \beta x + \cos \phi \sin \beta x)$, or $a \sin (\beta x + \phi)$.

Or, if we put $A/\sqrt{A^2 + B^2} = \cos \psi$ and $B/\sqrt{A^2 + B^2} = -\sin \psi$, our expression becomes $a(\cos \psi \cos \beta x - \sin \psi \sin \beta x)$, or $a \cos (\beta x + \psi)$.

or

$$e^{\alpha x}[(A_1 + A_2x) \cos \beta x + (B_1 + B_2x) \sin \beta x].$$

And in case such a pair occurs as r-fold roots, the corresponding part of the complementary function may be written

$$e^{\alpha x}[(A_1 + A_2x + \cdots + A_rx^{r-1}) \cos \beta x \\ + (B_1 + B_2x + \cdots + B_rx^{r-1}) \sin \beta x].$$

Exercise **1.** In the case of Ex. 2, §VI·2,

$\alpha = 3$, $\beta = 4$, so that the solution may also be written

$$y = e^{3x}(A \cos 4x + B \sin 4x),$$

or

$$y = ae^{3x} \sin (4x + \phi), \quad \text{or} \quad y = ae^{3x} \cos (4x + \psi).$$

2. $y'' + 2y = 0.$ **3.** $y^{\text{iv}} - y = 0.$

4. $(D^4 + 2D^2 + 1)y = 0.$ **5.** $(D^3 - D^2 + D)y = 0.$

REMARK. For the purpose of interpreting the solution of certain problems in Physics, it is desirable at times to introduce hyperbolic functions in place of the exponentials corresponding to two real roots of the auxiliary equation. Here we make use of the formulas $e^x = \cosh x + \sinh x$, $e^{-x} = \cosh x - \sinh x$.

First, suppose that two roots of the auxiliary equation are real and equal, except for the sign. Let them be m and $-m$. The corresponding terms of the complementary function are

$$c_1e^{mx} + c_2e^{-mx},$$

which may be written

$$(c_1 + c_2) \cosh mx + (c_1 - c_2) \sinh mx,$$

or

$$A \cosh mx + B \sinh mx.$$

Using the addition theorem for the hyperbolic functions, we can also write this in the form

$$a \cosh (mx + b), \quad \text{when } A^2 > B^2,$$
$$c \sinh (mx + d), \quad \text{when } A^2 < B^2.$$

Here a, b, c, d are arbitrary constants.

As an exercise, find the values of a and b, and of c and d, in terms of A and B, in their respective cases.

If, now, m_1 and m_2 are any two real roots of the auxiliary equation, we make use of the fact that two real numbers, k and l, exist such that

$$m_1 = k + l \quad \text{and} \quad m_2 = k - l.$$

Then

$$c_1e^{(k+l)x} + c_2e^{(k-l)x}$$

may be written

$$e^{kx}(c_1e^{lx} + c_2e^{-lx}),$$

or

$$e^{kx}(A \cosh lx + B \sinh lx).$$

VI·5. Particular integral. From what has gone before, it is clear that the problem of solving a complete linear differential equation with constant coefficients naturally resolves itself into first finding the complementary function and then finding a particular integral.

The former of these has been disposed of very satisfactorily. It has been shown to depend upon the solution of an algebraic equation.

Correspondingly, we shall show (§§VI·7, VI·8, VI·9) that the problem of finding a particular integral can be solved by n quadratures in the case of a linear differential equation of the nth order. In certain cases, these quadratures can be carried out readily, and such general methods prove entirely practical. In other cases, they are found to be cumbersome and unnecessary. Many special methods have been devised for avoiding them. In this treatise we shall present three general methods, each of which has its own practical advantage, and shall conclude with a special method, applicable to most cases that actually arise.

VI·6. Properties of the symbolic operator $(D - \alpha)$. In this discussion, α is a constant.

1° $(D - \alpha)y$ means $\dfrac{dy}{dx} - \alpha y$. Similarly, $(D - \beta)y$ means $\dfrac{dy}{dx} - \beta y$. Hence, $(D - \alpha)y + (D - \beta)y$ means $2\dfrac{dy}{dx} - (\alpha+\beta)y$, which may be written symbolically $[2D - (\alpha + \beta)]y$. That is, the result of operating on y with $(D - \alpha)$ and $(D - \beta)$ separately and then taking the sum is the same as operating on y with $2D - (\alpha + \beta)$. Hence we see that *the operation resulting from taking the sum of the results of two operations of the type here considered can be obtained symbolically by taking the sum of their symbolic representatives.* Thus, we can write

$$(D - \alpha) + (D - \beta) = 2D - (\alpha + \beta).$$

Evidently, this rule applies to the sum of any finite number of such operators, and also to the difference between any two of them.

2° $(D - \beta)(D - \alpha)y$ means $\left(\dfrac{d}{dx} - \beta\right)\left(\dfrac{dy}{dx} - \alpha y\right)$, which is $\dfrac{d^2y}{dx^2} - (\alpha + \beta)\dfrac{dy}{dx} + \alpha\beta y$. That is, the effect of operating first on y with $(D - \alpha)$, and then on the result with $(D - \beta)$, is the same as operating on y with $D^2 - (\alpha + \beta)D + \alpha\beta$. Hence we see that *the operation resulting from the successive performance of two operations of the type here considered can be obtained symbolically by taking the product of their symbolic representatives.* Thus, we can write

$$(D - \beta)(D - \alpha) = D^2 - (\alpha + \beta)D + \alpha\beta.$$

Moreover, owing to the symmetrical way in which α and β appear in the right-hand member, we see that *the order of the operators on the left is not essential*, or, as it is usually stated, *two operations of the type here considered are commutative.*

Obviously all this applies to any finite number of operations of the type here considered.

The results of this section can be summarized as follows:

The symbolic representatives of operations of the type here considered behave like algebraic quantities for the processes of addition, subtraction, and multiplication.

Remark. Since any polynomial in D with constant coefficients is a product of linear factors, this theorem applies also to operators whose symbolic representatives are polynomials in D with constant coefficients. Thus, if the roots of the auxiliary equation of a linear differential equation, with constant coefficients, are $m_1, m_2, \cdots, m_n$ (whether these are all distinct or not), we may write the differential equation in the form

$$k_0(D - m_1)(D - m_2) \cdots (D - m_n)y = X.$$

VI·7. Method of obtaining a particular integral. A general method for obtaining a particular integral of a complete linear differential equation with constant coefficients (and, for that matter, another method for obtaining the complementary function, as well) results from the following considerations:

Consider the differential equation of the third order

$$f(D)y \equiv (D - m_1)(D - m_2)(D - m_3)y = X, \tag{1}$$

in which $k_0 = 1$, and the m's are distinct.*

To find an integral of this equation we must determine a function of x, such that, when $f(D)$ operates on it, the result is X.

Let $(D - m_2)(D - m_3)y = u$, where u is a new function. Then (1) reduces to

$$(D - m_1)u = X, \quad \text{or} \quad \frac{du}{dx} - m_1 u = X.$$

This is a linear equation of the first order, and e^{-m_1x} is an integrating factor (§II·9). Hence,

$$e^{-m_1x}u = \int e^{-m_1x}X\,dx + c, \quad \text{or} \quad u = e^{m_1x}\int e^{-m_1x}X\,dx + ce^{m_1x};$$

i.e.,

$$(D - m_2)(D - m_3)y = e^{m_1x}\int e^{-m_1x}X\,dx + ce^{m_1x}.$$

---

* This is selected for simplicity, and yet is adequate to illustrate the general case.

Now let

$$(D - m_3)y = v.$$

Then

$$(D - m_2)v = e^{m_1x}\int e^{-m_1x}X\,dx + ce^{m_1x}.$$

This is also linear and of the first order; hence an integrating factor is e^{-m_2x}. Introducing this, we have

$$ve^{-m_2x} = \int e^{(m_1-m_2)x}\left[\int e^{-m_1x}X\,dx\right]dx + \frac{c}{m_1 - m_2}e^{(m_1-m_2)x} + c',$$

or

$$v = e^{m_2x}\int e^{(m_1-m_2)x}\left[\int e^{-m_1x}X\,dx\right]dx + \frac{c}{m_1 - m_2}e^{m_1x} + c'e^{m_2x}.$$

Hence,

$$(D - m_3)y = e^{m_2x}\int e^{(m_1-m_2)x}\left[\int e^{-m_1x}X\,dx\right]dx + c''e^{m_1x} + c'e^{m_2x},$$

where

$$c'' = \frac{c}{m_1 - m_2}.$$

This, too, is linear, and an integrating factor is e^{-m_3x}. Using this, we have

$$ye^{-m_3x} = \int e^{(m_2-m_3)x}\left\{\int e^{(m_1-m_2)x}\left[\int e^{-m_1x}X\,dx\right]dx\right\}dx$$
$$+ \frac{c''}{m_1 - m_3}e^{(m_1-m_3)x} + \frac{c'}{m_2 - m_3}e^{(m_2-m_3)x} + c_3,$$

or

$$y = e^{m_3x}\int e^{(m_2-m_3)x}\left\{\int e^{(m_1-m_2)x}\left[\int e^{-m_1x}X\,dx\right]dx\right\}dx$$
$$+ c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x}.$$

This suggests the following form of solution* when the equation is of the nth order and all the roots of the auxiliary equation are distinct:

$$\text{I.}\qquad y = e^{m_nx}\int e^{(m_{n-1}-m_n)x}\int \cdots \int e^{(m_1-m_2)x}\int e^{-m_1x}X(dx)^n$$
$$+ c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}.$$

---

* To prove this, we simply need assume it for the nth order, and show that it holds for the $(n + 1)$st order. This will be left as an exercise.

Remark. In the second line of I we have the complementary function, with which we are already familiar (§VI·2). (Let the student show that in the case of repeated roots of the auxiliary equation this method leads to the result found in §VI·3.) In the first line we have the particular integral, *whether the roots of the auxiliary equation are distinct or not.* Since the order of the factors in (1) is immaterial, it is desirable to take the m's in such an order as to simplify the successive quadratures that must be performed.

Exercise 1. $y''' - y'' - 2y' = e^{-x}$.

The auxiliary equation is

$$m^3 - m^2 - 2m = 0. \quad \therefore m = 0, -1, 2.$$

The complementary function is

$$Y = c_1 + c_2e^{-x} + c_3e^{2x}.$$

The particular integral is

$$\begin{aligned} V &= e^{2x}\int e^{(-1-2)x}\int e^{(0+1)x}\int e^{-x}(dx)^3 \\ &= e^{2x}\int e^{-3x}\int e^{x}(-e^{-x})(dx)^2 = -e^{2x}\int e^{-3x}\left[\int dx\right]dx \\ &= -e^{2x}\int e^{-3x}x\,dx = \tfrac{1}{3}xe^{-x} + \tfrac{1}{9}e^{-x}. \end{aligned}$$

Since e^{-x} is already part of the complementary function, it will be sufficient to use $\frac{1}{3}xe^{-x}$, thus giving the general solution

$$y = c_1 + c_2e^{-x} + c_3e^{2x} + \tfrac{1}{3}xe^{-x}.$$

If the m's are chosen in the order 0, 2, -1, the quadratures are simpler.

2. $y'' + 3y' + 2y = e^{e^x}$.

3. $(D^3 + 3D^2 + 3D + 1)y = 2e^{-x} - x^2e^{-x}$.

4. $(D^2 - D - 2)y = \sin x$. 5. $(D - 1)^2y = \dfrac{e^x}{(1 - x)^2}$.

VI·8. Another method of finding a particular integral.* The general method of finding the particular integral given in

---

* This was first published by Lobatto, *Théorie des Caractéristiques*, Amsterdam, 1837. Independently it was given by Boole, *Cambridge Mathematical Journal*, 1st series, Vol. II, p. 114.

§VI·7 is frequently long. At times, the first quadrature is readily obtained, but the successive ones are apt to be tedious. In this section we shall show how the operational method can be modified, so as to require only n simple quadratures when the roots of the auxiliary equation are distinct.

Starting with

$$(D - m_1)(D - m_2) \cdots (D - m_n)y = X,$$

we can write symbolically

$$y = \frac{1}{(D - m_1)(D - m_2) \cdots (D - m_n)} X,$$

where $\dfrac{1}{(D - m_1)(D - m_2) \cdots (D - m_n)}$ is the symbol of the operation inverse to $(D - m_1)(D - m_2) \cdots (D - m_n)$. Just as x results from operating on $\sin^{-1} x$ with the operator "sin," so, if we operate on

$$\frac{1}{(D - m_1)(D - m_2) \cdots (D - m_n)} X$$

with $(D - m_1)(D - m_2) \cdots (D - m_n)$, we obtain X. Moreover, we recall that the order in which the successive operators $(D - m_1)$, $(D - m_2), \cdots, (D - m_n)$ are employed is not essential.

Looked upon algebraically, the fraction

$$\frac{1}{(D - m_1)(D - m_2) \cdots (D - m_n)}$$

is equal to the sum of the partial fractions

$$\frac{a_1}{D - m_1} + \frac{a_2}{D - m_2} + \cdots + \frac{a_n}{D - m_n},$$

if the roots of the auxiliary equation are distinct.

Looked upon as operators, this equality still holds; for, to verify the equality, we operate on both with $(D - m_1)(D - m_2) \cdots (D - m_n)$. Since the order in which we operate with these factors is immaterial, it follows that all the operators resulting from operating on the sum of partial fractions are polynomials, which can be treated as algebraic expressions. Hence the algebraic equality of the symbolic representatives of the two resulting operators means the equality of these operators. Since the operator $(D - m_1)(D - m_2) \cdots (D - m_n)$ is not ambiguous, it follows that the original operators, in fractional form, are also equivalent; i.e.,

$$y = \frac{1}{(D - m_1)(D - m_2) \cdots (D - m_n)} X$$

$$\equiv \frac{a_1}{D - m_1} X + \frac{a_2}{D - m_2} X + \cdots + \frac{a_n}{D - m_n} X.$$

If we put $u = \dfrac{a}{D - m} X$, then $(D - m)u = aX$.

Integrating this linear equation, we have $ue^{-mx} = a\int e^{-mx} X dx$, or $u = ae^{mx}\int e^{-mx}X dx$. Hence the particular integral may be put in the form

$$\text{II.}\quad a_1 e^{m_1 x}\int e^{-m_1 x} X dx + a_2 e^{m_2 x}\int e^{-m_2 x} X dx + \cdots + a_n e^{m_n x}\int e^{-m_n x} X dx.$$

In case a root is repeated, the following obvious modification is necessary:

To illustrate, we shall suppose that one of the roots, m_1, is a triple root. The corresponding partial fractions will be $\dfrac{a_1}{D - m_1} + \dfrac{a_2}{(D - m_1)^2} + \dfrac{a_3}{(D - m_1)^3}$, and the corresponding terms of the particular integral will be (I, §VI·7)

$$a_1 e^{m_1 x}\int e^{-m_1 x} X dx + a_2 e^{m_1 x}\int\int e^{-m_1 x} X (dx)^2 + a_3 e^{m_1 x}\int\int\int e^{-m_1 x} X (dx)^3.$$

Exercise **1.** $(D^2 - 3D + 2)y = e^x$.

2. $(D^3 - 3D^2 - D + 3)y = x^2$. **3.** $y''' - 4y'' + 5y' - 2y = x$.

VI·9. Variation of parameters.* Another general method of obtaining the particular integral, known as the method of

---

* This method is due to Joseph Louis Lagrange (1736–1813).

variation of parameters, applies quite readily at times, especially if the order of the equation is not high. The method consists in replacing the constants in the complementary function by undetermined functions of x, such that, when the modified expression for the complementary function is substituted for y in $f(D)y$, we obtain X, and not zero.

Since we have n functions at our disposal and only one condition to impose upon them, it is clear that we can satisfy this requirement in an indefinite number of ways, by imposing upon them any $n - 1$ additional conditions consistent with the first one. In actual practice, we shall impose these conditions in such a way that only first derivatives of the undetermined functions appear.

The method will be carried out in the case of an equation of the third order. (The argument will readily be seen to apply to any order.)

Let the equation be

$$(1) \qquad f(D)y \equiv (k_0D^3 + k_1D^2 + k_2D + k_3)y = X,$$

and let its complementary function be

$$c_1y_1 + c_2y_2 + c_3y_3.$$

The method consists in finding three functions of x, viz. v_1, v_2, v_3, such that

$$(2) \qquad y = v_1y_1 + v_2y_2 + v_3y_3$$

shall be a solution of (1). This still allows us to impose two additional conditions upon v_1, v_2, v_3.

Differentiating (2), we have

$$Dy = v_1y_1' + v_2y_2' + v_3y_3' + y_1v_1' + y_2v_2' + y_3v_3'.$$

We shall now use one of the two conditions at our disposal by letting

$$(3) \qquad y_1v_1' + y_2v_2' + y_3v_3' = 0;$$

then

$$(4) \qquad Dy = v_1y_1' + v_2y_2' + v_3y_3'.$$

Differentiating this, we have

$$D^2y = v_1y_1'' + v_2y_2'' + v_3y_3'' + y_1'v_1' + y_2'v_2' + y_3'v_3'.$$

Here, again, we put

$$y_1'v_1' + y_2'v_2' + y_3'v_3' = 0, \tag{5}$$

thus using the second condition still at our disposal, so that

$$D^2y = v_1y_1'' + v_2y_2'' + v_3y_3''. \tag{6}$$

Differentiating again, we have

$$D^3y = v_1y_1''' + v_2y_2''' + v_3y_3''' + y_1''v_1' + y_2''v_2' + y_3''v_3'. \tag{7}$$

Substituting the values given by (2), (4), (6), and (7) in (1), we have

$$v_1f(D)y_1 + v_2f(D)y_2 + v_3f(D)y_3 + k_0(y_1''v_1' + y_2''v_2' + y_3''v_3') = X.$$

Since y_1, y_2, y_3 are solutions of $f(D)y = 0$, this reduces to

$$k_0y_1''v_1' + k_0y_2''v_2' + k_0y_3''v_3' = X. \tag{8}$$

Since y_1, y_2, y_3 are three linearly independent solutions of the homogeneous linear differential equation obtained from (1) by putting $X = 0$, it follows (compare Note II in the Appendix) that the three equations (3), (5), and (8) are independent. They can, therefore, be solved for v_1', v_2' and v_3'. By quadratures, values of v_1, v_2, v_3 may then be found which, when substituted in (2), give the general solution of (1). The terms containing the constants of integration, arising from these quadratures, give the complementary function again.

It is important to note that *the method of variation of parameters applies in the case of all linear differential equations, whether the coefficients are constants or not.* The other two methods, given previously, for obtaining the particular integral (§§VI·7, VI·8) are restricted to the case of constant coefficients.

As an illustration, we shall solve the general linear differential equation of the first order (§II·9),

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{9}$$

by the method of variation of parameters.

Consider, first, the homogeneous linear differential equation

$$\frac{dy}{dx} + Py = 0,$$

in which the variables are separable. Its solution is

$$y = ce^{-\int P dx}.$$

Our problem is to find a function v, so that

$$y = ve^{-\int P dx} \tag{10}$$

shall satisfy (9). Differentiating, we obtain

$$\frac{dy}{dx} = -vPe^{-\int P dx} + e^{-\int P dx}\frac{dv}{dx}.$$

Substituting in (9), we see that (10) will be a solution of the latter, if

$$e^{-\int P dx}\frac{dv}{dx} = Q, \quad \text{or} \quad \frac{dv}{dx} = Qe^{\int P dx};$$

i.e., if

$$v = \int Qe^{\int P dx} dx + C.$$

Hence we have the general solution of (9) in the form

$$y = e^{-\int P dx}\left[\int Qe^{\int P dx} dx + C\right].$$

Compare this with the form found in §II·9.

***Exercise* 1.** $y'' + y = \sec x$.

The roots of the auxiliary equation are $\pm i$. Hence the complementary function is $c_1 \cos x + c_2 \sin x$. Employing the method of variation of parameters, we have

$$\begin{aligned} y &= v_1 \cos x + v_2 \sin x, \\ Dy &= -v_1 \sin x + v_2 \cos x + \Bigg| \cos x \frac{dv_1}{dx} + \sin x \frac{dv_2}{dx} = 0, \\ D^2y &= -v_1 \cos x - v_2 \sin x - \sin x \frac{dv_1}{dx} + \cos x \frac{dv_2}{dx}. \end{aligned}$$

Substituting in the differential equation, we have

$$-\sin x\frac{dv_1}{dx}+\cos x\frac{dv_2}{dx}=\sec x.$$

Besides, we chose

$$\cos x\frac{dv_1}{dx}+\sin x\frac{dv_2}{dx}=0.$$

$$\therefore \frac{dv_1}{dx}=-\sin x\sec x=-\frac{\sin x}{\cos x}, \qquad v_1=\log\cos x+C_1.$$

$$\frac{dv_2}{dx}=1, \qquad v_2=x+C_2.$$

Therefore, the complete solution is

$$y=C_1\cos x+C_2\sin x+\cos x\log\cos x+x\sin x.$$

2. $\dfrac{d^2y}{dx^2}+y=\tan x.$ **3.** $y''+y=\cot^2 x.$

4. $y''-y=\dfrac{2}{e^x-1}.$ **5.** $y'''+y'=\tan x.$

6. $(D-1)^2y=\dfrac{e^x}{(1-x)^2}.$ (Ex. 5, §VI·7.)

VI·10. Method of undetermined coefficients. We shall conclude the discussion of the problem of finding a particular integral with an account of a method, which, while not applicable in all cases, is relatively simple whenever it can be used. It applies to all cases in which the right-hand member contains only terms which have a finite number of linearly independent derivatives. Such terms are x^h, e^{lx}, $\sin mx$, $\cos nx$, and products of these, where h is any positive integer and l, m, n are any constants.

If we take as a trial particular integral, V, the terms of the right-hand member X, each prefixed by an undetermined constant multiplier, we shall find that, in general, on substituting this in the left-hand member of the complete linear differential equation, other terms arise as a result of differentiation. Consequently, we shall use for V the sum of all the terms of X, together with all functions arising from them by differentiation and forming with them a linearly independent set (by hypothesis there is only a finite number of these), each prefixed by an undetermined constant multiplier. We substitute this expression for y in the differential equation, and equate the result identically to X (i.e., we equate the coefficients of corresponding terms). This gives rise to a set of equations of condition among the undetermined multipliers, from which the latter can frequently be calculated. The following exercises will illustrate:

***Exercise* 1.** $y'' + 4y = x^2 + \cos x$.

The roots of the auxiliary equation are $\pm 2i$.

$$\therefore Y = A \cos 2x + B \sin 2x.$$

For a particular integral we take

$$V = ax^2 + bx + c + f \cos x + g \sin x.$$

Then

$$D^2 V = 2a - f \cos x - g \sin x.$$

$$\therefore f(D) V = 4ax^2 + 4bx + 2a + 4c + 3f \cos x + 3g \sin x.$$

Equating the coefficients in the right-hand member to the corresponding ones in X, we have

$$\begin{aligned} 4a &= 1 & \therefore a &= \tfrac{1}{4}, \\ 4b &= 0 & b &= 0, \\ 2a + 4c &= 0 & c &= -\tfrac{1}{8}, \\ 3f &= 1 & f &= \tfrac{1}{3}, \\ 3g &= 0 & g &= 0. \end{aligned}$$

Hence the general solution is

$$y = A \cos 2x + B \sin 2x + \tfrac{1}{4}x^2 - \tfrac{1}{8} + \tfrac{1}{3} \cos x.$$

***Exercise* 2.** $(D^2 - 2D + 1)y = 2xe^{2x} - \sin^2 x$.

The complementary function is readily seen to be

$$Y = (c_1 + c_2 x)e^x.$$

To find the particular integral, it will be simpler to replace $-\sin^2 x$ by $\frac{1}{2} \cos 2x - \frac{1}{2}$. The right-hand member then takes the form

$$2xe^{2x} + \tfrac{1}{2} \cos 2x - \tfrac{1}{2}.$$

As a trial particular integral we shall now take

$$V = axe^{2x} + be^{2x} + c \cos 2x + f \sin 2x + g.$$

$$\therefore DV = 2axe^{2x} + (a + 2b)e^{2x} + 2f \cos 2x - 2c \sin 2x.$$

$$D^2 V = 4axe^{2x} + 4(a + b)e^{2x} - 4c \cos 2x - 4f \sin 2x.$$

$$f(D) V = axe^{2x} + (2a + b)e^{2x} - (3c + 4f) \cos 2x - (3f - 4c) \sin 2x + g.$$

Hence we must have

$$\begin{aligned} a &= 2 & \therefore\quad a &= 2, \\ 2a + b &= 0 & b &= -4, \\ 3c + 4f &= -\tfrac{1}{2} & & \\ 3f - 4c &= 0 & c &= -\tfrac{3}{50},\quad f = -\tfrac{2}{25}, \\ g &= -\tfrac{1}{2} & g &= -\tfrac{1}{2}, \end{aligned}$$

and the general solution is

$$y = (c_1 + c_2x)e^x + 2xe^{2x} - 4e^{2x} - \tfrac{3}{50}\cos 2x - \tfrac{2}{25}\sin 2x - \tfrac{1}{2}.$$

3. $(D^2 + 1)y = 2e^x + x^3 - x.$

4. $(D^2 + 2D + 1)y = 3e^{2x} - \cos x.$ **5.** $(D^3 - 1)y = x^2.$

This method fails in either of the two following cases:

1° If a term in the right-hand member is also a term in the complementary function, it is clear that the result of substituting such a term or any of its derivatives for y in $f(D)y$ is zero. This follows from the definition of the complementary function, and from the fact that the derivative of a term in the complementary function is also a term in it.

We shall first suppose that the root of the auxiliary equation, to which the term in question corresponds, is a simple one; if u is the term, we have

$$f(D)u = 0, \quad \text{but } f'(D)u \neq 0.$$

Now, since $f(D)$ is a polynomial in D, and since

$$D^k(xu) = xD^ku + kD^{k-1}u,$$

it is clear that

$$\begin{aligned} f(D)(xu) &= xf(D)u + f'(D)u \\ &= f'(D)u \neq 0. \end{aligned}$$

That $f'(D)u$ actually contains the term u, in this case, may be seen at once, if u is taken in the form e^{mx}, whether m is real or not; for, $f'(D)e^{mx} = f'(m)e^{mx}$, and $f'(m) \neq 0$, when m is a simple root of the auxiliary equation.

If u is taken in the form $e^{\alpha x}\cos\beta x$, it is possible that $f'(D)(e^{\alpha x}\cos\beta x)$ may not contain a term in $e^{\alpha x}\cos\beta x$. But, since $f'(D)u$ is not identically zero, $f'(D)(e^{\alpha x}\cos\beta x)$ must then contain a term in $e^{\alpha x}\sin\beta x$, and, therefore, $f'(D)(e^{\alpha x}\sin\beta x)$ will contain a term in $e^{\alpha x}\cos\beta x$. This follows from the fact that $\sin\beta x = \cos(\beta x - \pi/2)$. Similarly, if $u = e^{\alpha x}\sin\beta x$ and $f'(D)(e^{\alpha x}\sin\beta x)$ does not contain a term in $e^{\alpha x}\sin\beta x$, it must contain one in $e^{\alpha x}\cos\beta x$; and $f'(D)(e^{\alpha x}\cos\beta x)$ will then contain one in $e^{\alpha x}\sin\beta x$, as may be seen by writing $\cos\beta x = \sin(\beta x + \pi/2)$. Moreover, no other terms can arise.

Thus, we see that the substitution of $xu +$ linearly independent terms derived from it by differentiation (each term multiplied by an undetermined constant) for y in $f(D)y$ gives rise to a set of terms including u. Moreover, all the other terms of the set are obtained by differentiating u.

If, now, u is a term in the complementary function, which corresponds to an r-fold root of the auxiliary equation,

$$f(D)u = 0,\ f'(D)u = 0,\ \cdots,\ f^{(r-1)}(D)u = 0, \text{ but } f^{(r)}(D)u \neq 0.$$

Since

$$D^k(x^r u) = x^r D^k u + krx^{r-1}D^{k-1}u + \frac{k(k-1)}{2}r(r-1)x^{r-2}D^{k-2}u + \cdots + \frac{k(k-1)\ldots(k-s+1)}{s!}r(r-1)\cdots(r-s+1)x^{r-s}D^{k-s}u + \cdots + k(k-1)\cdots(k-r+1)D^{k-r}u,\text{*}$$

and since $f(D)$ is a polynomial in D with constant coefficients, it follows that

$$f(D)(x^r u) = x^r f(D)u + rx^{r-1}f'(D)u + \frac{r(r-1)}{2}x^{r-2}f''(D)u + \cdots + \frac{r(r-1)\cdots 3\cdot 2}{(r-1)!}xf^{(r-1)}(D)u + f^{(r)}(D)u.$$

All the terms on the right are zero, except $f^{(r)}(D)u$, which is definitely not zero. As before, it can be shown that, if for y in $f(D)y$ we substitute $x^r u +$ linearly independent terms arising from this by differentiation (each term multiplied by an undetermined constant), we obtain the term u together with such others only as arise from it by differentiation.

---

* This follows readily from Leibniz's theorem for the derivative of a product, and is not restricted to the special forms for u, appearing in the complementary function. For example, see the author's *Calculus*, §47.

2° The second case where the original method will fail arises when terms of the type $x^t u$ occur, u being a term in the complementary function. A similar modification of the method applies here. Suppose that u corresponds to an r-fold root of the auxiliary equation.* As before, we have

$$\begin{aligned} f(D)(x^{t+r}u) = x^{t+r}f(D)u &+ (t+r)x^{t+r-1}f'(D)u \\ &+ \frac{(t+r)(t+r-1)}{2!}x^{t+r-2}f''(D)u + \cdots \\ &+ \frac{(t+r)(t+r-1)\cdots(t+2)}{(r-1)!}x^{t+1}f^{(r-1)}(D)u \\ &+ \frac{(t+r)(t+r-1)\cdots(t+1)}{r!}x^{t}f^{(r)}(D)u + \cdots. \end{aligned}$$

The term involving $f^{(r)}(D)u$ is the first one on the right which is definitely not zero. Hence, as before, it can be shown that, if for y in $f(D)y$ we substitute $x^{t+r}u$ + linearly independent terms arising from it by differentiation (each term multiplied by an undetermined constant), we obtain $x^t u$ together with such other terms only as arise from it by differentiation.

We are now in a position to formulate our rule:

When the right-hand member of the differential equation contains only terms which have a finite number of linearly independent derivatives, take for trial particular integral the sum of all the terms together with all those obtained from them by differentiation and linearly independent of them, prefixing to each of them an undetermined constant coefficient. These coefficients are determined† by substituting the trial particular integral in the differential equation and equating coefficients.

In case the right-hand member contains a term $x^t u$, where t is a positive integer or zero and u is a term in the complementary

---

* Of course, $xu, x^2u, \cdots, x^{r-1}u$ are also terms in the complementary function. In this discussion, u is supposed to be that term which does not contain x as a factor.

† For a proof of the fact that the equations of condition on the undetermined constants, obtained by equating coefficients, are consistent and sufficient to determine the values of the undetermined constants, see A. B. Coble, *American Mathematical Monthly*, Vol. 26 (1919), pp. 12–15.

function corresponding to an r-fold root of the auxiliary equation, replace that term in the right-hand member by $x^{t+r}u$ *in making up the trial particular integral.*

Remark. It may not always be necessary to insert all the terms suggested by the general rule. These can frequently be detected by inspection. Thus, in Ex. 1, since the coefficient of y' in the differential equation is zero, the terms x and $\sin x$ in the trial particular integral are unnecessary, for these will obviously not appear as a result of substituting $ax^2 + f\cos x$ in the equation. If any unnecessary terms are used in the trial particular integral, their coefficients will, of course, be found to be zero. On the other hand, it is also obviously unnecessary to employ any terms which appear in the complementary function. (If such a term is included in the trial particular integral, its coefficient will, of course, not appear in the resulting equations among the coefficients. This means that this coefficient may be chosen arbitrarily; which is exactly as it should be.)

6. $(D^3 - 2D^2 - 3D)y = 3x^2 + \sin x.$

Hint. Since 0 is a simple root of the auxiliary equation, 1 is a term in the complementary function. Hence, we shall have to try $ax^3 + bx^2 + cx$ to get x^2. Moreover, $\sin x$ is not a part of the complementary function. Hence the trial integral is

$$V = ax^3 + bx^2 + cx + f\sin x + g\cos x.$$

7. $(D^4 - 2D^2 + 1)y = e^x + 4.$

8. $(D^2 - 2D)y = e^{2x} + e^x + x + 3.$

9. $(D^4 + 2D^2 + 1)y = \cos x.$

Note. Many special rules are known for obtaining a particular integral corresponding to certain types of terms in the right-hand member. The method of this section leads to a number of these rules. The following list is given, not as additional methods to be memorized, but as exercises, to be established by the student, using the methods of this section:

(*a*) If a is not a root of the auxiliary equation, $V = e^{ax}/f(a)$, corresponding to the term e^{ax} in X.

(*b*) If a is an r-fold root of the auxiliary equation, $V = x^r e^{ax}/f^{(r)}(a)$, corresponding to the term e^{ax} in X.

(*c*) If $f(D)$ contains only even powers of D, write it $F(D^2)$. In case a is not a root of the auxiliary equation,

$V = \cos ax/F(-a^2)$ corresponding to the term $\cos ax$ in X,
$V = \sin ax/F(-a^2)$ corresponding to the term $\sin ax$ in X.

VI·11. Cauchy's linear equation. The linear differential equation

$$(1) \qquad k_0 x^n y^{(n)} + k_1 x^{n-1} y^{(n-1)} + \cdots + k_{n-1} x y' + k_n y = X,$$ [*]

where the coefficient of $y^{(r)}$ is a constant times x^r, is readily reducible by the transformation $x = e^u$ to an equation with constant coefficients. For,

$$\begin{aligned} y' &= \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{x}\frac{dy}{du}, \\ y'' &= \frac{dy'}{dx} = \qquad\qquad = \frac{1}{x^2}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right), \\ y''' &= \frac{dy''}{dx} = \qquad\qquad = \frac{1}{x^3}\left(\frac{d^3y}{du^3} - 3\frac{d^2y}{du^2} + 2\frac{dy}{du}\right). \end{aligned}$$

. . . .

If we write $\frac{dy}{du} = \mathfrak{D}y$, we have

$$\begin{aligned} xDy &= \mathfrak{D}y, \\ x^2D^2y &= \mathfrak{D}(\mathfrak{D}-1)y, \\ x^3D^3y &= \mathfrak{D}(\mathfrak{D}-1)(\mathfrak{D}-2)y, \\ &\cdots\cdots\cdots\cdots\cdots \\ x^nD^ny &= \mathfrak{D}(\mathfrak{D}-1)(\mathfrak{D}-2)\cdots(\mathfrak{D}-n+1)y, \end{aligned}$$ [†]

---

[*] This form of linear equation is sometimes called the homogeneous linear equation. This seems rather unfortunate. I prefer to reserve this name for any linear equation which is homogeneous in y and its derivatives, in conformity with a large number of writers on the subject, and shall refer to the above linear equation (1) as the *Cauchy linear differential equation*, after Augustin Louis Cauchy (1789–1857). See his *Exercises d'Analyse*, Vol. I, p. 262.

[†] This general formula of transformation may be established by assuming that

$$y^{(n)} = \frac{1}{x^n}\mathfrak{D}(\mathfrak{D}-1)\cdots(\mathfrak{D}-n+1)y,$$

and (1) becomes

$$(2)\ [k_0\mathfrak{D}(\mathfrak{D}-1)\cdots(\mathfrak{D}-n+1)+k_1\mathfrak{D}(\mathfrak{D}-1)\cdots(\mathfrak{D}-n+2) + \cdots + k_{n-1}\mathfrak{D}+k_n]y = U,$$

where U is the form assumed by X after the transformation.

Equation (2) is obviously also linear, but with constant coefficients.

More generally, the equation

$$k_0(a+bx)^n y^{(n)} + k_1(a+bx)^{n-1}y^{(n-1)} + \cdots + k_{n-1}(a+bx)y' + k_n y = X^*$$

is readily seen to be reducible to a linear equation with constant coefficients by the substitution $a + bx = e^u$.

***Exercise* 1.** $x^3y''' + xy' - y = x \log x$.

If we let $x = e^u$, this becomes

$$[\mathfrak{D}(\mathfrak{D}-1)(\mathfrak{D}-2)+\mathfrak{D}-1]y = ue^u,$$

or

$$(\mathfrak{D}^3 - 3\mathfrak{D}^2 + 3\mathfrak{D} - 1)y = ue^u.$$

The roots of the auxiliary equation are 1, 1, 1.

Hence the complementary function is $Y = (c_1 + c_2u + c_3u^2)e^u$.

In this case, method I (§VI·7) gives the particular integral at once. We have

$$V = e^u\int\int\int e^{-u}ue^u(du)^3 = e^u\frac{u^4}{24}.$$

$\therefore$ The solution is $y = (c_1 + c_2u + c_3u^2)e^u + \dfrac{u^4e^u}{24}$,

or

$$y = [c_1 + c_2\log x + c_3(\log x)^2]x + \frac{x(\log x)^4}{24}.$$

2. $(x^3D^3 + 2x^2D^2 + 2)y = 10\left(x + \dfrac{1}{x}\right)$.

---

and showing that differentiation with respect to x leads to

$$y^{(n+1)} = \frac{1}{x^{n+1}}\mathfrak{D}(\mathfrak{D}-1)\cdots(\mathfrak{D}-n)y.$$

Since the formula holds for $n = 2$, it holds for n any positive integer. This method of establishing the formula was suggested to me by Dr. Teresa Cohen.

* This form of the linear equation is referred to as *Legendre's linear equation*, after Adrien Marie Legendre (1752–1833).

3. $(x^2D^2 + 3xD + 1)y = \dfrac{1}{(1 - x)^2}$.

4. $(x + 1)^2y'' - 4(x + 1)y' + 6y = x$.

5. $(x + 2)^2y''' + (x + 2)y'' + y' = 1$. [Multiply by $x + 2$.]

VI·12. Summary. The problem of solving a linear differential equation consists of two parts, the finding of the complementary function, and the finding of a particular integral (§VI·1).

The finding of the complementary function in the case of an equation with constant coefficients $f(D)y = X$ depends only upon the solution of the algebraic equation $f(m) = 0$. According as the roots are distinct and real, repeated, or complex, the complementary function takes one of the forms indicated in §§VI·2, VI·3, VI·4. Four methods for finding the particular integral are given in §§VI·7, VI·8, VI·9, VI·10.

An estimate of the relative merits of these methods may be summarized as follows: The methods of §§VI·7 and VI·8 (which will be referred to as I and II respectively) and that of variation of parameters (§VI·9) have the advantage of generality, the last one applying also to linear equations in which the coefficients may be functions of the independent variable. But, as is usually true in such cases, the carrying out of these general methods may be tedious. The method of variation of parameters has the great advantage of being readily retained in mind, but is frequently long and laborious, especially if the equation is of higher order than the second. The method of undetermined coefficients (§VI·10), although not general, applies to a very large number of cases that actually occur. In those cases where it does apply, it has the advantage of involving only the operations of differentiation and the solution of simultaneous linear algebraic equations. Integration is not required. Besides, it is very readily retained in mind. The actual work of carrying out this method is straightforward and not difficult. It may at times be long, but usually it is no longer than the other methods, if as long.

As a rule, then, whenever the method of undetermined coefficients applies, it is probably the most desirable one to use. An instance of an exception to this is illustrated by Ex. 1, §VI·11. [Method I is always preferable in case the differential equation is of the form

$$(D - \alpha)^n y = e^{\alpha x} F(x),$$

where $F(x)$ can be integrated readily n times consecutively.]

Method II is of relatively little practical use. It has been given primarily because of its applications in the theory.

We noted in §VI·I, 2° that

$$F(D)(V_1 + V_2) = F(D)V_1 + F(D)V_2.$$

Hence, if it is obvious on inspection that different methods apply most readily to the various terms in the right-hand member, it is advisable to find the particular integral for any term or group of terms by the method best adapted, and then take the sum of these. This is true, for instance, in the case of Ex. 3 below.

The equation

$$k_0(a+bx)^n y^{(n)} + k_1(a+bx)^{n-1}y^{(n-1)} + \cdots + k_{n-1}(a+bx)y' + k_n y = X$$

(including as a special case the Cauchy equation where $a = 0$, $b = 1$) is reducible to one with constant coefficients by the substitution $a + bx = e^u$ (§VI·11).

Exercise **1.** $(D^2 - 5D + 6)y = \cos x - e^{2x}$.

2. $(D^4 - 1)y = e^x \cos x$.

3. $(D^2 + 2D + 1)y = 2x^3 - xe^{3x} + e^{-x}\sin x$.

4. $(D + 1)^3 y = xe^{-x} - e^x$. **5.** $(D^3 - 4D)y = x^2 - 3e^{2x}$.

6. $(D^4 - 2D^2 + 1)y = \cos x$.

7. $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)y = (1 + \log x)^2$.

8. $(D^3 + 2D^2 + D)y = x^2 - x + x^{-1} - 2x^{-2} + 2x^{-3}$.

9. $(D^2 + 4)y = \sin^2 x$. **10.** $(D^2 + 1)y = \sec^2 x$.

11. $(D - 1)^3 y = x - x^3e^x$.

12. $(D^4 - D^3 - 3D^2 + 5D - 2)y = e^{3x}$.

13. $(D^2 + 1)y = x \cos x.$

14. $(x^3D^3 + 2x^2D^2 - xD + 1)y = 1/x.$

15. $(D^3 - 1)y = xe^x + \cos^2 x.$

16. $(D - 1)^2y = \cos x + e^x + x^2e^x.$

VI·13. Applications. The solution of many important problems of Physics involves linear differential equations with constant coefficients. Some of these will be considered here.

1° ***Simple harmonic motion.*** This type of motion arises when a particle, free to move in a straight line, is subject to an attracting force, which is directed toward a fixed point O on the line of motion and is proportional to the distance of the particle from O. It is approximately the motion involved in the case of the swing of a simple pendulum or of an undamped galvanometer needle through a very small arc, the oscillation of a point on the prong of a tuning fork, or the vibration up and down of a weighted particle suspended by a coiled spring. We shall consider here the case of the *simple pendulum* of length l and mass m swinging in a vacuum.

The only acting force is gravity; it acts vertically downward, and its intensity is $-mg$. If s represents the length of arc measured from the lowest point of the pendulum, then at any moment when the pendulum makes an angle θ with the vertical, $s = l\theta$, and the acceleration is $l\dfrac{d^2\theta}{dt^2}$. The component of the force of gravity along the tangent to the path is $-mg \sin \theta$. Hence the equation of motion is

$$ml\frac{d^2\theta}{dt^2} = -mg \sin \theta.$$

If θ remains very small throughout the motion, we may replace $\sin \theta$ by θ, as a first approximation. Our equation then takes the form

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

This is the *differential equation of simple harmonic motion.* Solving this, we have the *equation of simple harmonic motion*

$$\theta = A \cos \mu t + B \sin \mu t, \tag{1}$$

where $\mu^2 = g/l$. As noted before, §VI·4, an alternate form for this is

$$\theta = a \cos (\mu t + b). \tag{2}$$

Here a is the amplitude of the swing and b determines the phase. In any particular experiment, the values of these are fixed when the values of θ and $d\theta/dt$ are known at a given instant. These are the *boundary conditions* or the *initial conditions* of the given problem.

The *period* of the swing is defined as the time that elapses between two consecutive instants at which the pendulum occupies the same position (such as that of equilibrium) and is moving in the same direction. From (2), we see that t_1 and t_2 will be such consecutive instants if $\mu(t_2 - t_1) = 2\pi$. Hence the period is equal to $2\pi/\mu$ or $2\pi\sqrt{l/g}$. It is determined by the nature of the apparatus and does not depend upon the initial conditions.

***Exercise* 1.** Find the values of a and b, and the resulting form of (2), if the pendulum is pulled aside so that $\theta = \theta_0$, kept at rest there for a short time, and then allowed to swing.

Here the initial conditions are

$$\theta = \theta_0, \quad \frac{d\theta}{dt} = 0 \quad \text{when } t = 0. \tag{3}$$

Differentiating (2), we have

$$\frac{d\theta}{dt} = -a\mu \sin (\mu t + b). \tag{4}$$

Using (3) in (2) and (4), we have

$$\theta_0 = a \cos b \quad \text{and} \quad 0 = -a\mu \sin b. \tag{5}$$

Since $\theta_0 \neq 0$, it follows that $a \neq 0$; and evidently $\mu \neq 0$. Hence the second equation of (5) leads to

$$\sin b = 0. \quad \therefore b = 0.$$

Using this in the first equation of (5), we find that

$$a = \theta_0;$$

whence

$$\theta = \theta_0 \cos \mu t.$$

Exercise **2.** Find the values of a and b, and the resulting form of (2) if, while the pendulum is at rest, it is given a sudden blow so that it starts off with an angular velocity $d\theta/dt = \omega_0$, and is then permitted to swing.

Here the initial conditions are

$$\theta = 0, \quad \frac{d\theta}{dt} = \omega_0 \quad \text{when } t = 0.$$

Using these in (2) and (4), we have

$$0 = a \cos b \quad \text{and} \quad \omega_0 = -a\mu \sin b. \tag{6}$$

Since $\omega_0 \neq 0$, $a \neq 0$; whence we conclude from the first equation of (6) that

$$\cos b = 0. \quad \therefore b = \frac{\pi}{2}.$$

Using this in the second equation of (6), we find that

$$a = -\frac{\omega_0}{\mu};$$

whence

$$\theta = \frac{\omega_0}{\mu} \sin \mu t.$$

3. Find the form of (2) when the initial conditions are

$$\theta = \theta_0, \quad \frac{d\theta}{dt} = \omega_0 \quad \text{when } t = 0.$$

4. Find the values of A and B, and the resulting form of (1) when the initial conditions are those of Ex. 1, 2, and 3 respectively.

2° ***Damped vibration.*** If the motion takes place in a resisting medium where the resisting force is proportional to the velocity, the differential equation of motion (to a first approximation) is also linear with constant coefficients. If the resisting force is taken as $-2km\dfrac{ds}{dt}$ or $-2kml\dfrac{d\theta}{dt}$, the differential equation of motion is

$$ml\frac{d^2\theta}{dt^2} = -mg\theta - 2kml\frac{d\theta}{dt},$$

or

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \mu^2\theta = 0,$$

where, as before, $\mu^2 = g/l$. The roots of the auxiliary equation are

$$-k \pm \sqrt{k^2 - \mu^2}. \tag{7}$$

The relative values of k and μ play an important rôle in the form of the solution.

(*a*) If $k < \mu$, i.e., if the resisting force is relatively weak, the roots (7) are imaginary. Putting $k^2 - \mu^2 = -\lambda^2$, we may write the equation of motion in the form

$$\theta = e^{-kt}(A \cos \lambda t + B \sin \lambda t), \tag{8}$$

or

$$\theta = ae^{-kt} \cos (\lambda t + b). \tag{9}$$

The amplitude is ae^{-kt}, which diminishes continually, approaching zero asymptotically.

The period is $2\pi/\lambda$, a constant, but larger than $2\pi/\mu$, the period in the case of motion in a vacuum.

(*b*) If $k = \mu$, the equation of motion is

$$\theta = (c_1 + c_2t)e^{-kt}. \tag{10}$$

(*c*) If $k > \mu$, the equation of motion is

$$\theta = e^{-kt}(c_1e^{\nu t} + c_2e^{-\nu t}), \tag{11}$$

where $\nu^2 = k^2 - \mu^2$.

***Exercise* 5.** Find the forms of (8) and (9) when the initial conditions are those of Ex. 1, 2, and 3 respectively.

6. Find the forms of (10) using the initial conditions of Ex. 1 and 2. Show that, in each case, θ is always of the same sign, diminishing continually in absolute value and approaching zero asymptotically.

7. Show that, with the initial conditions of Ex. 3, (10) takes the form

$$\theta = [\theta_0 + (\omega_0 + k\theta_0)t]e^{-kt}.$$

Discuss the motion when θ_0 and $\omega_0 + k\theta_0$ have opposite signs, also when these have the same sign; in the latter case, consider what happens when θ_0 and ω_0 have the same sign, and when they have opposite signs.

8. Find the form of (11), using the initial conditions of Ex. 3. Discuss possible types of motion in this case for various values of θ_0 and ω_0.

3° *Forced vibration.* If an external periodic force is introduced in case 1° or case 2° (*a*), the additional motion superinduced thereby is known as *forced vibration.* The differential equations of motion in these cases are

$$\frac{d^2\theta}{dt^2} + \mu^2\theta = C \cos pt \tag{12}$$

and

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \mu^2\theta = C \cos pt, \tag{13}$$

respectively.

It will be left as an exercise to show that the particular integral due to the presence of the right-hand member of (12), i.e., the term expressing the forced vibration, is

$$\frac{C}{\mu^2 - p^2} \cos pt, \text{ if } p \neq \mu, \tag{14}$$

and

$$\frac{Ct}{2\mu} \sin \mu t, \text{ if } p = \mu. \tag{15}$$

In the case of (13), it is

$$\frac{C[(\mu^2 - p^2) \cos pt + 2kp \sin pt]}{(\mu^2 - p^2)^2 + 4k^2p^2}. \tag{16}$$

If we put

$$l = +\sqrt{(\mu^2 - p^2)^2 + 4k^2p^2},\ \cos\alpha = (\mu^2 - p^2)/l,\ \sin\alpha = 2kp/l,$$

this may be written in the form

$$\frac{C}{l}\cos(pt - \alpha). \tag{17}$$

If $p = \mu$, then $\alpha = \pi/2$, and both (16) and (17) take the form

$$\frac{C}{2k\mu}\sin\mu t. \tag{18}$$

The forced vibration given in (14) and (17) is simple harmonic, with the period $2\pi/p$, which is the same as that of the applied force. Moreover, it is not damped. In the case of damped vibration, the amplitude of the free vibration, given by (9), the complementary function of (13), diminishes continually. Hence the forced vibration becomes dominating after a while. This will occur soon when k is very small and p is nearly equal to μ, for then the constant amplitude, C/l, of the forced vibration is large. This amplitude will be a maximum when p is such as to make l a minimum. Equating the derivative of l to zero, we find this value of p to be $\sqrt{\mu^2 - 2k^2}$. Then $l = 2k\lambda$, and the maximum amplitude is $C/2k\lambda$.

In this case the applied periodic force is said to be *in resonance* with the vibrating body. Here the period of the forced vibration, $2\pi/p \equiv 2\pi/\sqrt{\mu^2 - 2k^2}$, is approximately the same as that of the free vibration, $2\pi/\lambda \equiv 2\pi/\sqrt{\mu^2 - k^2}$.

This phenomenon of *resonance* has important applications in Acoustics and other branches of Physics. Use is made of this principle, for example, in wireless transmission and in the seismograph, an apparatus for registering earth tremors.

When $p = \mu$, we see from (18) that the phase of the forced vibration is behind that of the impressed force by $\pi/2$, in the case of damped vibration. In the simple harmonic case, (15) shows that the amplitude increases indefinitely with t. Of course, the differential equation of motion was obtained by assuming certain limits to the amplitude of the swing. Hence, the argument ceases to hold after these limits are reached. However, serious injury to the apparatus may result if the experiment is prolonged.

Exercise 9. Find the particular integral of

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \mu^2\theta = Ce^{-kt}\cos pt,$$

where the impressed force is also damped. Distinguish between the cases when $p \neq \lambda = \sqrt{\mu^2 - k^2}$ and when $p = \lambda$.

4° ***Distribution of temperature along a wire.*** Consider a wire suspended in air, the temperature of which is supposed to be constant. There will be no loss of generality in assuming a scale of temperatures on which that of the air is zero. Suppose, now, that one end of the wire is maintained at a constant temperature θ_0. Heat will flow along the wire, and, in addition, at each point along the wire, heat will be lost to the surrounding air. After a time, the rates of both this flow along the wire and the loss at each point will become constant. When this steady state has been attained, what is the temperature at each point along the wire?

If we consider a small length of wire, extending from x to $x + dx$,* the amount of heat flowing into it by conduction is

$$-ka\left(\frac{d\theta}{dx}\right)_x, \tag{19}$$

where k is a constant depending upon the material of which the wire is made, a (or πr^2, where r is the radius of the cross-section of the wire) is the area of the cross-section, and $\left(\frac{d\theta}{dx}\right)_x$ is the rate of change of temperature along the wire at the point x. The $-$ sign is used because $\left(\frac{d\theta}{dx}\right)_x$ is negative.

---

* Here we use the differential dx instead of the increment Δx because we shall ignore higher powers of the former.

The amount of heat flowing out of the small length from x to $x + dx$ consists of two parts:

First, the amount that flows along the wire at the point $x + dx$, which is

$$-ka\left(\frac{d\theta}{dx}\right)_{x+dx}, \quad \text{or} \quad -ka\left[\left(\frac{d\theta}{dx}\right)_x + \left(\frac{d^2\theta}{dx^2}\right)_x dx + \cdots\right],$$

where $\cdots$ stands for terms involving higher powers of dx.

Second, the amount lost to the surrounding air, which is approximately proportional to the product of the exposed area of the wire and θ, the difference in the temperatures of the wire and air. The latter is taken to be constant along the small length. If the circumference of the wire is $c = 2\pi r$, this amount is

$$lc\theta dx,$$

where l is the factor of proportionality. Hence the amount of heat flowing out of the small length is

$$(20) \qquad -ka\left(\frac{d\theta}{dx}\right)_x - ka\left(\frac{d^2\theta}{dx^2}\right)_x dx + \cdots + lc\theta dx.$$

Since the flow is steady, the quantities (19) and (20) must be equal. Hence the distribution of temperature is given by the differential equation*

$$ka\frac{d^2\theta}{dx^2} - lc\theta = 0,$$

or

$$(21) \qquad \frac{d^2\theta}{dx^2} - \mu^2\theta = 0,$$

where $\mu^2 = lc/ka = 2l/kr$.

The general solution of (21) is

$$(22) \qquad \theta = c_1e^{\mu x} + c_2e^{-\mu x}, \quad \text{or} \quad \theta = A \cosh \mu x + B \sinh \mu x.$$

Exercise 10. Find the expression for θ, (*a*) if the wire is of length L and the constant temperature at the far end is θ_1; (*b*) if the wire is very long, so that L may be taken as practically infinite, and θ_1 as practically zero. In both cases $\theta = \theta_0$ when $x = 0$.

---

* Since $d^2\theta/dx^2$ means precisely the same thing as $(d^2\theta/dx^2)_x$, there is no need of retaining the latter notation.

5° ***Distribution of temperature in a hollow sphere.*** Consider a hollow sphere surrounded by air, whose temperature is constant. Suppose that a constant source of heat, radiating uniformly in every direction, is applied at the common center of the inner and outer surfaces of the hollow sphere. The temperature of the inner surface is raised, and heat flows radially through the solid to the outer surface, and then passes on to the air. After a while the flow becomes steady; i.e., the temperature, θ, at each point of the solid is constant, being a function of r, the distance of the point from the common center of the inner and outer surfaces of the hollow sphere. Our problem is to find an expression for this function.

Consider the flow of heat through a spherical shell, whose inner and outer surfaces are concentric with the hollow sphere, and have the radii r and $r + dr$ respectively. As before, we use the differential dr because we shall ignore higher powers of it.

The amount of heat passing across the inner surface of the shell is

$$-4\pi r^2 k\left(\frac{d\theta}{dr}\right)_r, \tag{23}$$

where $4\pi r^2$ is the area of the inner surface, k is a constant depending upon the material of which the solid is made, and $\left(\frac{d\theta}{dr}\right)_r$ is the rate of change of θ along a radius of the sphere at a distance r from the center.

The amount of heat passing across the outer boundary of the shell is

$$-4\pi(r + dr)^2 k\left(\frac{d\theta}{dr}\right)_{r+dr},$$

or

$$-4\pi(r + dr)^2 k\left[\left(\frac{d\theta}{dr}\right)_r + \left(\frac{d^2\theta}{dr^2}\right)_r dr + \cdots\right],$$

where $\cdots$ stands for terms involving higher powers of dr. Expanding and neglecting higher powers of dr than the first, we find that this reduces to

$$(24)\qquad -4\pi k\left[r^2\left(\frac{d\theta}{dr}\right)_r+\left\{2r\left(\frac{d\theta}{dr}\right)_r+r^2\left(\frac{d^2\theta}{dr^2}\right)_r\right\}dr\right].$$

Since there is no other flow of heat, (23) and (24) must be equal. Equating them, and dropping the subscript r in the case of $\left(\frac{d\theta}{dr}\right)_r$ and $\left(\frac{d^2\theta}{dr^2}\right)_r$, since the unaffected derivatives have precisely the same significance, we have the differential equation

$$(25)\qquad r^2\frac{d^2\theta}{dr^2}+2r\frac{d\theta}{dr}=0.$$

Its solution is

$$(26)\qquad \theta=\frac{c_1}{r}+c_2.$$

Exercise **11.** If the temperature of the inner surface of the hollow sphere is θ_0 and that of the outer surface is θ_1, and the radii of these two surfaces are r_0 and r_1 respectively, determine the values of the arbitrary constants in the general solution (26). What values do these assume when the scale of temperatures is so chosen that, on it, $\theta_1 = 0$? Also find the quantity of heat flowing across an isothermal spherical surface of radius r, where $r_0 < r < r_1$, taking $\theta_1 = 0$.

6° ***Distribution of temperature in a very long hollow cylinder.*** Consider a hollow circular cylinder surrounded by air, whose temperature is constant. Suppose that a constant source of heat is applied uniformly along the common axis of the inner and outer surfaces of the hollow cylinder. The temperature of the inner surface is raised. If the cylinder is long enough, the heat flows outward through parts of the cylinder, sufficiently far from the ends, along straight lines, perpendicular to the axis of the cylinder, to the outer surface, and then passes on to the air. After a while the flow becomes steady; i.e., the temperature, θ, at each point of the solid is constant, being a function of r, the distance of the point from the axis of the cylinder. Our problem is to find an expression for this function.

Consider the flow of heat through a cylindrical shell, with the same axis as the hollow cylinder, the radii of the inner and outer surfaces of the shell being r and $r + dr$ respectively. Again, we use the differential dr here, because we shall ignore higher powers of it. In addition, we shall consider a part of the shell near the middle and of length l, which is very small compared with the entire length of the cylinder.

The amount of heat passing across the inner surface of the shell is

$$(27) \qquad -2\pi rlk\left(\frac{d\theta}{dr}\right)_r,$$

where $2\pi rl$ is the area of the inner surface, k is a constant depending upon the material of which the solid is made, and $\left(\frac{d\theta}{dr}\right)_r$ is the rate of change of θ along a line perpendicular to the axis, at a distance r from the latter.

The amount of heat passing across the outer surface of the shell is

$$-2\pi(r + dr)lk\left(\frac{d\theta}{dr}\right)_{r+dr},$$

or

$$-2\pi(r + dr)lk\left[\left(\frac{d\theta}{dr}\right)_r + \left(\frac{d^2\theta}{dr^2}\right)_r dr + \cdots\right],$$

where $\cdots$ stands for terms involving higher powers of dr. Expanding and neglecting higher powers of dr than the first, we find that this reduces to

$$(28) \qquad -2\pi rlk\left(\frac{d\theta}{dr}\right)_r - 2\pi lk\left[\left(\frac{d\theta}{dr}\right)_r + r\left(\frac{d^2\theta}{dr^2}\right)_r\right] dr.$$

Since there is no other flow of heat through this piece of shell, (27) and (28) must be equal. Equating them, and dropping the subscript r in the case of $\left(\frac{d\theta}{dr}\right)_r$ and $\left(\frac{d^2\theta}{dr^2}\right)_r$, we have the differential equation

$$r\frac{d^2\theta}{dr^2} + \frac{d\theta}{dr} = 0. \tag{29}$$

Its solution is

$$\theta = c_1 + c_2 \log r. \tag{30}$$

***Exercise* 12.** If the temperature of the inner surface of the hollow cylinder is θ_0, and that of the outer surface is 0, and the radii of a right section of these surfaces are r_0 and r_1 respectively, determine the values of the arbitrary constants in (30). Also find the quantity of heat flowing across a unit length of an isothermal cylindrical surface at a distance r from the axis, where $r_0 < r < r_1$.

7° *Electric circuit in a condenser.* If a condenser of capacity C, charged with a quantity of electricity Q, is introduced into an electric circuit, it will discharge by sending a current through the circuit. If q is the quantity of electricity in the condenser at any instant during the discharge, it satisfies the equation

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0,$$

where L is the self-inductance and R the resistance of the circuit.

Here the auxiliary equation is

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0.$$

$$\therefore m_1, m_2 = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}.$$

1. If $R^2 > 4L/C$, we have

$$q = Ae^{m_1 t} + Be^{m_2 t}.$$

To determine A and B, we use the initial conditions that when $t = 0$, $q = Q$ and $-dq/dt \equiv i = 0$, where i is the current; i.e.,

$$A + B = Q \quad \text{and} \quad m_1A + m_2B = 0;$$

whence

$$A = -\frac{m_2Q}{m_1 - m_2}, \; B = \frac{m_1Q}{m_1 - m_2}.$$

Hence

$$q = \frac{Q}{m_1 - m_2}(m_1 e^{m_2 t} - m_2 e^{m_1 t}),$$

and

$$i = -\frac{dq}{dt} = \frac{m_1 m_2 Q}{m_1 - m_2}(e^{m_1 t} - e^{m_2 t}).$$

Noting the values of m_1 and m_2, we see that q and i diminish continually, but do not become zero for a finite value of t. However, they soon become practically negligible, when $R/2L$ is a large quantity.

2. If $R^2 = 4L/C$, we have

$$q = e^{-Rt/2L}(A + Bt).$$

To determine A and B, we make use of the same initial conditions as before, and find

$$q = \frac{Q}{2L}(2L + Rt)e^{-Rt/2L},$$

and

$$i = -\frac{dq}{dt} = \frac{QR^2 t}{4L^2}e^{-Rt/2L}.$$

Here, again, q and i diminish rapidly, without vanishing for a finite value of t. However, as a rule, they are soon negligible.

3. If $R^2 < 4L/C$,

$$m_1,\, m_2 = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \alpha \pm i\beta.$$

$$\therefore q = e^{\alpha t}(A \cos \beta t + B \sin \beta t),$$

and

$$i = -\frac{dq}{dt} = -e^{\alpha t}[(\alpha A + \beta B)\cos \beta t + (\alpha B - \beta A)\sin \beta t].$$

To determine A and B, we use the same initial conditions as before, and find

$$q = \frac{Qe^{\alpha t}}{\beta}(\beta \cos \beta t - \alpha \sin \beta t),$$

and

$$i = \frac{Q(\alpha^2 + \beta^2)}{\beta} e^{\alpha t} \sin \beta t,$$

where $\alpha = -R/2L$ and $\beta = \sqrt{1/LC - R^2/4L^2}$.

Both q and i are oscillating functions of period $T = 2\pi/\beta$. Thus, they are sometimes positive and sometimes negative. The amplitude in either case is a constant times $e^{-Rt/2L}$, which usually diminishes very rapidly with t. But, in specially constructed circuits in which R is small relative to L, an oscillatory discharge may be realized.

8° ***Central force.*** Consider the motion of a particle subjected to an attractive force, $F(\rho)$, always directed toward a fixed point and a function of the distance of the particle from that point. Take this point as the origin. The components of the force acting upon the particle when located at the point (x, y) are

$-F \cos \theta$ or $-Fx/\rho$ in the direction of the x-axis,
$-F \sin \theta$ or $-Fy/\rho$ in the direction of the y-axis.

Here ρ and θ are the polar coördinates of the point (x, y).

If the mass of the particle is taken as unity, the equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{Fx}{\rho} \quad \text{and} \quad \frac{d^2y}{dt^2} = -\frac{Fy}{\rho}. \tag{31}$$

Multiplying these by y and x respectively, and taking their difference, we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0. \tag{32}$$

The left-hand member is the derivative of $x\dfrac{dy}{dt} - y\dfrac{dx}{dt}$, as may be verified readily. Hence, after integrating, we have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h, \tag{33}$$

where h is a constant.

Since $\theta = \tan^{-1}(y/x)$,

$$\frac{d\theta}{dt} = \frac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2 + y^2} = \frac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{\rho^2}.$$

Hence equation (33) may be written

$$\rho^2 \frac{d\theta}{dt} = h. \tag{34}$$

This equation has an interesting interpretation.

The differential of the area swept over by the radius vector as the particle moves from (ρ, θ) to $(\rho + d\rho, \theta + d\theta)$ is equal to $\frac{1}{2}\rho^2 d\theta$. Hence equation (34) shows that *the rate at which the radius vector sweeps over an area is constant.** If this area is designated by A, equation (34) may be written

$$\frac{dA}{dt} = \frac{h}{2}.$$

Integrating, we have

$$A = \frac{ht}{2} + c.$$

If we begin to measure A when $t = 0$, i.e., if we impose the initial condition that $A = 0$ when $t = 0$, we have

$$A = \frac{ht}{2}.$$

To find the differential equation of the path, we make the change of variable

$$\rho = \frac{1}{u}. \tag{35}$$

Then equation (34) becomes

$$\frac{d\theta}{dt} = hu^2. \tag{36}$$

---

* This is one of Kepler's laws of planetary motion about the sun.

Moreover, $x = \cos\theta/u$, whence

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = hu^2\frac{dx}{d\theta} = hu^2\left(-\frac{\sin\theta}{u} - \frac{\cos\theta}{u^2}\frac{du}{d\theta}\right)$$

$$= -h\left(u\sin\theta + \cos\theta\frac{du}{d\theta}\right),$$

$$\frac{d^2x}{dt^2} = \frac{d}{d\theta}\left(\frac{dx}{dt}\right)\frac{d\theta}{dt} = hu^2\frac{d}{d\theta}\left(\frac{dx}{dt}\right)$$

$$= -h^2u^2\cos\theta\left(u + \frac{d^2u}{d\theta^2}\right).$$

Noting that the first equation of (31) may also be written

$$\frac{d^2x}{dt^2} = -F(\rho)\cos\theta,$$

we have the differential equation of the path in the form

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) - F\left(\frac{1}{u}\right) = 0. \tag{37}$$

[Of course, the same differential equation results from making the change of variable (35) in the second equation of (31).]

Here θ, the independent variable, is absent.

***Exercise* 13.** Solve equation (37) if $F(1/u) = ku^2$; that is, if the force is *inversely proportional to the square of the distance of the particle from the center.*

In this case, the differential equation reduces to the linear one with constant coefficients

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2}.$$

It will be left as an exercise to show that the general solution of this is

$$u = c\cos(\theta - \alpha) + \frac{k}{h^2}, \tag{38}$$

where c and α are arbitrary constants, to be determined by the initial conditions.

If we replace c by ke/h^2, where e is now the arbitrary constant, equation (38) takes the form

$$u = \frac{k}{h^2}[1 + e\cos(\theta - \alpha)]. \tag{39}$$

Passing back to the variable ρ, we find the equation of the path, in polar coördinates, to be

$$\rho = \frac{\frac{h^2}{k}}{1 + e\cos(\theta - \alpha)}. \tag{40}$$

We recognize this to be the equation of a conic, with a focus at the origin (or pole),* with the axis of the conic through the focus making an angle α with the initial line. The eccentricity of the conic is equal to e, and the semi-latus rectum is equal to h^2/k.

To determine the constants of integration in a particular case, we shall suppose that, when $\theta = 0$, the particle is at a distance ρ_0 from the center of the force and is moving with a speed v_0 in a direction making an angle ψ_0 with the initial line. If the angle which the tangent to the curve makes with the radius vector to the point of contact is ψ, we know that $d\rho/d\theta = \rho\cot\psi$. [Compare formula ($q$) §III·2.] Hence our initial conditions are

$$\rho = \rho_0, \quad \frac{d\rho}{d\theta} = \rho_0\cot\psi_0, \quad \text{when } \theta = 0,$$

or

$$u = \frac{1}{\rho_0}, \quad \frac{du}{d\theta} = -\frac{\cot\psi_0}{\rho_0}, \quad \text{when } \theta = 0.$$

Putting these in equation (39) and in

$$\frac{du}{d\theta} = -\frac{ke}{h^2}\sin(\theta - \alpha),$$

we have

$$\frac{1}{\rho_0} = \frac{k}{h^2}(1 + e\cos\alpha) \quad \text{and} \quad \frac{\cot\psi_0}{\rho_0} = -\frac{ke}{h^2}\sin\alpha,$$

or

$$e\sin\alpha = -\frac{h^2\cot\psi_0}{\rho_0 k} \quad \text{and} \quad e\cos\alpha = \frac{h^2}{\rho_0 k} - 1.$$

---

* This is another of Kepler's laws of planetary motion about the sun.

Hence

$$\tan \alpha = \frac{h^2 \cot \psi_0}{\rho_0 k - h^2} \tag{41}$$

and

$$e^2 = \frac{h^4 \csc^2 \psi_0}{\rho_0{}^2 k^2} - \frac{2h^2}{\rho_0 k} + 1. \tag{42}$$

The value of e, the eccentricity of the conic, determines the nature of the conic. To investigate this, we make use of the formula for the distance of the tangent to a curve from the origin, in polar coördinates, (w) §III·2,

$$p = \rho^2 \frac{d\theta}{ds}.$$

From (34), we have

$$h = \rho^2 \frac{d\theta}{dt} = \rho^2 \frac{d\theta}{ds} \frac{ds}{dt} = pv.$$

Here, v is the velocity of the particle. Moreover, $p = \rho \sin \psi$. Hence

$$h = \rho v \sin \psi.$$

Since h is constant, we have

$$h = \rho_0 v_0 \sin \psi_0.$$

Putting this in (42), we find, after a simple computation, that

$$1 - e^2 = \frac{\rho_0{}^2 v_0{}^2 \sin^2 \psi_0}{k^2} \left(\frac{2k}{\rho_0} - v_0{}^2 \right). \tag{43}$$

Neglecting the case when $\sin \psi_0 = 0$, i.e., when $\psi_0 = 0$ or π,* we see that the value of ψ_0 plays no rôle in determining the type of conic. From (43) we see that, when $\sin \psi_0 \neq 0$,

the path is an ellipse if $v_0{}^2 < 2k/\rho_0$,
the path is a parabola if $v_0{}^2 = 2k/\rho_0$,
the path is an hyperbola if $v_0{}^2 > 2k/\rho_0$.

In particular, if $\psi_0 = \pi/2$ and $v_0{}^2 = k/\rho_0$, e equals zero and the path is a circle.

***Exercise* 14.** Solve equation (37) if $F(1/u) = ku^3$; that is, if the force is *inversely proportional to the cube of the distance.*

---

* In this case we have motion in a straight line, considered in Ex. 22, §VIII·10.

In this case also, the differential equation reduces to a linear one with constant coefficients, viz.,

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{k}{h^2}\right)u = 0.$$

Distinguish between the cases when $1 - k/h^2$ is greater than zero and when it is less than zero.

9° *Additional exercises leading to the solution of linear differential equations with constant coefficients.*

***Exercise* 15.** Find the motion of a heavy particle sliding without friction along a massless straight rod, which rotates about one of its points in a vertical plane with constant angular velocity ω.

Consider, first, the motion of the particle along the rod. Here the acceleration of the particle arises from two sources, viz., the weight of the particle and the rotation of the rod.

The acceleration along the rod, due to gravity, is equal to $-g \sin \omega t$, if we suppose the rod to be horizontal when $t = 0$.

The acceleration along the rod, due to the rotation of the rod, is equal to $\omega^2 r$, where r is the distance of the particle from the center of rotation.

Hence the differential equation of motion along the rod is

$$\frac{d^2r}{dt^2} - \omega^2 r = -g \sin \omega t.$$

Find the solution satisfying the initial conditions $r = r_0$ and v (or dr/dt) $= v_0$ when $t = 0$.

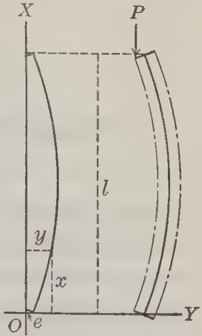
For what values of r_0 and v_0 does the motion along the rod become simple harmonic?

***Exercise* 16.** In the case of an elastic string, whose length is l, and which has one end fixed and a weighted particle attached to the other end, the equation of motion of the particle is

$$\frac{d^2s}{dt^2} = -\frac{g}{e}(s - l),$$

where e is the elongation due to the particle. Find s and v (or ds/dt), assuming that $s = s_0$ and $v = 0$ when $t = 0$.

***Exercise* 17.** If a vertical column is subjected to a load whose center of gravity is directly over the central line of the column, a compression of the column results. But, if it is at a horizontal distance e from this central line, there is a tendency to bend the column. Both ends are free to turn about a horizontal axis, but not free to move laterally. See figure.



If we take the line of the applied force as the axis of x and the origin at the lower end, the differential equation to determine the horizontal deflection of points along the central line is (as in the case of the deflection of beams, Ex. 18, §III·4)

$$\text{curvature} \equiv \frac{y''}{(1 + y'^2)^{3/2}} = \frac{M}{EI}.$$

As before, since y' is very small, we neglect the term y'^2 as a first approximation, and consider the differential equation

$$EIy'' = M.$$

If the weight of the load is P, the bending moment with respect to any point (x, y) is

$$M = -Py.$$

Hence the differential equation to be solved is the linear one with constant coefficients

$$EIy'' + Py = 0,$$

with the initial or boundary conditions $y = e$ when $x = 0$, $y' = 0$ when $x = l/2$, the length of the column being taken as l.

Find the values of y and y' at various points along the central line. Also find their greatest values.

CHAPTER VII

LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

VII·1. Change of dependent variable. While the problem of solving linear differential equations of the first order can always be reduced to one of quadratures (§II·9), there is no general method by which the solution of linear differential equations of higher order can be so effected.

In this short chapter three methods, especially applicable to linear differential equations of the second order, are developed. By the first of these an equation of special type can be transformed into a linear differential equation of the first order; the other two enable one to transform certain equations of special types into linear ones with constant coefficients.

While these methods apply to only a relatively small number of differential equations that are apt to arise, it is very easy, as a rule, to test whether any of them is applicable in any given case. Hence little time need be consumed with them if they are found not to be applicable in any particular problem.

The general type of a linear equation of the second order is

$$y'' + Py' + Qy = X, \tag{1}$$

where P, Q, X are functions of x only.

Let us try the following change of dependent variable,

$$y = y_1 v, \tag{2}$$

where v is the new dependent variable and y_1 is a function of x still to be so determined as to simplify the resulting differential equation (3). Then

$$y' = y_1 v' + y_1' v, \quad y'' = y_1 v'' + 2y_1' v' + y_1'' v;$$

and equation (1) becomes

$$(3) \qquad v'' + P_1v' + Q_1v = X_1,$$

where

$$P_1 = \frac{2y_1'}{y_1} + P, \quad Q_1 = \frac{y_1'' + Py_1' + Qy_1}{y_1}, \quad X_1 = \frac{X}{y_1}.$$

Two uses may be made of this:

1° By inspection,* or other means, a particular integral may be found for the corresponding homogeneous differential equation

$$(4) \qquad y'' + Py' + Qy = 0,$$

which results from replacing X by zero. If this particular integral is taken as y_1 in (2), $Q_1 = 0$, and equation (3) becomes

$$v'' + P_1v' = X_1.$$

This may be written as the linear equation of the first order in v'

$$\frac{dv'}{dx} + P_1v' = X_1,$$

which can be solved for v' (§II·9). A quadrature will then give y.

Exercise 1. $y'' - x^2y' + xy = x.$

Here x is a particular integral, since $P \equiv -Qx$. Putting $y = xv$, we have

$$xv'' + (2 - x^3)v' = x,$$

or

$$\frac{dv'}{dx} + \left(\frac{2}{x} - x^2\right)v' = 1.$$

An integrating factor is $e^{\int(2/x - x^2)dx}$, or $x^2e^{-x^3/3}$.

$$\therefore v'x^2e^{-x^3/3} = \int x^2e^{-x^3/3}dx = -e^{-x^3/3} + c_1,$$

---

* Thus, for example, if $P \equiv -Qx$, x is evidently such a particular integral. Again, if $1 + P + Q \equiv 0$, e^x is such, or, if $1 - P + Q \equiv 0$, e^{-x} is one; or, more generally, it may be possible to note, by inspection, a number m, such that $m^2 + Pm + Q \equiv 0$; in this case e^{mx} is such an integral.

or

$$v' = -\frac{1}{x^2} + c_1 x^{-2} e^{x^3/3},$$

and

$$v = \frac{1}{x} + c_1 \int x^{-2} e^{x^3/3} dx + c_2;$$

whence

$$y = 1 + c_1 x \int x^{-2} e^{x^3/3} dx + c_2 x.$$

As noted in the case of Ex. 3, §II·1, the term involving the quadrature, which cannot be expressed in terms of elementary functions, may be replaced by an infinite series. It is suggested that the student verify that this term is equal to

$$c_1\left(-1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{5 \cdot 3^2 \cdot 2!} + \frac{x^9}{8 \cdot 3^3 \cdot 3!} + \frac{x^{12}}{11 \cdot 3^4 \cdot 4!} + \cdots\right).$$

2. $xy'' - (2x + 1)y' + (x + 1)y = x^2 - x - 1.$
3. $(1 + x^2)y'' + 2xy' - 2y = 3x.$
4. $(1 - x)y'' + xy' - y = (1 - x)^2.$

Here x and e^x are particular integrals, when the right-hand member is replaced by zero. Hence, by property A, §VI·1, the complementary function is $c_1x + c_2e^x$. To find the particular integral which must be added to the complementary function, the method of variation of parameters (§VI·9) may be employed.

5. $xy'' - (2x - 1)y' + (x - 1)y = x^2 - 2.$
6. $xy'' + (2x - 1)y' + (x - 1)y = x^3 + 3x^2 - 2x + 2.$
7. $(x^3 - 2x^2)y'' - (x^3 + 2x^2 - 6x)y' + (3x^2 - 6)y = 2x^3 - 2x^2.$

[Make use of the fact that a particular integral of the form x^m exists for the corresponding homogeneous differential equation.]

2° If no particular integral of the homogeneous differential equation (4) is known, there is sometimes an advantage in using the change of variable (2) in order to make $P_1 = 0$ in equation (3), i.e.,

$$\frac{2y_1'}{y_1} + P = 0.$$

For this we must take

$$y_1 = e^{-\int (P/2)dx}. \tag{5}$$

Upon differentiating, we find

$$y_1' = -(P/2)y_1 \quad \text{and} \quad y_1'' = -(P'/2)y_1 + (P/2)^2 y_1.$$

Putting these values in the expression for Q_1, we find

$$Q_1 = Q - \frac{d}{dx}\left(\frac{P}{2}\right) - \left(\frac{P}{2}\right)^2. \tag{6}$$

If $Q - P'/2 - (P/2)^2$ is equal to a constant, equation (3) is a linear differential equation with constant coefficients; if it is equal to a constant divided by x^2, equation (3) is a Cauchy equation (§VI·11). If Q_1 assumes neither of these forms, the method is not applicable.

Attention should be called to the fact that the value of Q_1 does not depend upon that of y_1. It is given by (6) in terms of the coefficients of the original differential equation. Thus, it is possible to determine whether the method is applicable or not without calculating y_1.

Exercise 8. $\sin x \cdot y'' + 2 \cos x \cdot y' + 3 \sin x \cdot y = e^x.$

Dividing by the coefficient of y'', we have

$$y'' + 2 \cot x \cdot y' + 3y = e^x \csc x.$$

Here $Q = 3$, $P/2 = \cot x.$ Hence

$$\begin{aligned} Q_1 &= Q - P'/2 - (P/2)^2 \\ &= 3 + \csc^2 x - \cot^2 x = 4, \end{aligned}$$

and the method is applicable. Now, we calculate y_1, which is

$$y_1 = e^{-\int (P/2)dx} = e^{-\int \cot x dx} = \csc x;$$

and, from the form of (3), we know that the transformation $y = v \csc x$ reduces our differential equation to

$$v'' + 4v = e^x.$$

Solving this, we find

$$v = c_1 \cos 2x + c_2 \sin 2x + e^x/5;$$

whence, after simple reductions,

$$y = c_1(\cos x \cot x - \sin x) + c_2 \cos x + \csc x \cdot e^x/5.$$

9. $y'' - 2 \tan x \cdot y' - (a^2 + 1)y = \sin x.$

10. $x^2y'' - 4x^3y' + 2(2x^2 + 1)(x^2 - 1)y = 2xe^{x^2}.$

11. $xy'' + 2y' - xy = 2e^x.$

12. $x^2y'' - 2x^2y' + (x^2 - 6)y = x^5 - 6x^4.$

VII·2. Change of independent variable. If we introduce a new independent variable u, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = u'\frac{dy}{du},$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2} = u'^2\frac{d^2y}{du^2} + u''\frac{dy}{du},$$

and the differential equation

$$y'' + Py' + Qy = X$$

becomes

$$\frac{d^2y}{du^2} + \frac{u'' + Pu'}{u'^2}\frac{dy}{du} + \frac{Q}{u'^2}y = \frac{X}{u'^2}. \tag{1}$$

It may happen that the coefficient of dy/du in (1) reduces to a constant when we put $Q/u'^2 = \pm 1$, i.e., $u' = \sqrt{\pm Q}$ (where we choose that sign which will make the square root real). If such is the case, the equation (1) is linear with constant coefficients. But, if this is not the case, the method is not applicable. Here, also, we note that the test for applicability can be made without calculating u.

Exercise **1.** $y'' + (2e^x - 1)y' + e^{2x}y = e^{4x}.$

Putting $u' = e^x$, we find $\dfrac{u'' + Pu'}{u'^2} = 2$, and the method is applicable. Hence, if we introduce the new variable $u = e^x$, the differential equation becomes

$$\frac{d^2y}{du^2} + 2\frac{dy}{du} + y = u^2.$$

Its solution is

$$y = (c_1 + c_2u)e^{-u} + u^2 - 4u + 6.$$

Replacing u by its value in terms of x, we have finally

$$y = (c_1 + c_2e^x)e^{-e^x} + e^{2x} - 4e^x + 6.$$

2. $y'' + \tan x \cdot y' + \cos^2 x \cdot y = 0.$

3. $(1 - x^2)y'' - xy' + 4y = 3x + 8x^2.$

4. $x^6y'' + 3x^5y' + y = 1/x^2.$

5. $xy'' - (2x^2 + 1)y' - 8x^3y = 4x^3e^{-x^2}.$

VII·3. Summary. There is no general method for solving all linear differential equations of the second order, $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$, by means of quadratures, or by other elementary methods. In this chapter several special types of linear differential equations of the second order are discussed, which are simplified by certain transformations of variables.

1° If, by inspection or otherwise, we know a particular integral y_1 when the right-hand member is made zero, then $y = y_1v$ will reduce the equation to a linear one of the first order when dv/dx is taken to be a new variable (§VII·1, 1°).

2° If such a particular integral is not known, the value of $Q - P'/2 - (P/2)^2$ should be found. If this is a constant, or a constant divided by x^2, the equation is reducible to one with constant coefficients, or to a Cauchy equation, by substituting $y = y_1v$, where $y_1 = e^{-\int (P/2)dx}$ (§VII·1, 2°).*

3° If the previous method does not apply, let $u' = \sqrt{\pm Q}$ (using that sign which will make the square root real); then $(u'' + Pu')/u'^2$ should be calculated. If this is equal to a constant, the method of §VII·2 applies.*

Exercise **1.** $xy'' - (x + 3)y' + 3y = 2x - 3.$

2. $(x - 3)y'' - (4x - 9)y' + (3x - 6)y = 0.$

3. $x^2y'' + 4xy' + (2 - x^2)y = 0.$

---

* Emphasis should be laid on the fact that no integration is required in testing the applicability of this method. It is only after one is assured that the method applies, that the form of the new variable need be sought.

4. $(x^2 + 1)y'' - 2xy' + 2y = 4.$

5. $x^2y'' - xy' - 4x^4y = 2 + 4x^4 \log x.$

6. $x^2y'' - 4xy' + (6 + x^2)y = x^5.$

7. $xy'' - (3x^2 + 1)y' - 4x^3y = 4x^5.$

8. $x^2y'' - 2x(1 + x)y' + 2(1 + x)y = x^3.$

9. $x^2y'' - 2nxy' + (n^2 + n + a^2x^2)y = 0.$

10. $x^4y'' + 2x^3y' + n^2y = 0.$ **11.** $x^4y'' - xy' + y = 0.$

[Express the solution of Ex. 11 in the form of an infinite series in negative, integral powers of x.]

12. $(x^2 + 2x)y'' + (x^2 - 2)y' - 2(x + 1)y = (2x^3 + 6x^2)e^{-x}$

CHAPTER VIII

MISCELLANEOUS METHODS FOR SOLVING EQUATIONS OF HIGHER ORDER THAN THE FIRST

VIII·1. General plan of solution. There is no general elementary method for solving a differential equation of higher order than the first, excepting in the case of linear equations with constant coefficients and equations reducible to these (Chapter VI). A common procedure, in other cases, consists in transferring the problem to that of solving an equation of lower order. We shall consider some classes of equations for which this can be done.

VIII·2. Dependent variable absent. If y is absent, the equation is of the form

$$f(x, y', y'', \cdot \cdot \cdot, y^{(n)}) = 0.$$

Putting $y' = p$, and noting that

$$y'' = \frac{dp}{dx} \equiv p', y''' = p'', \cdot \cdot \cdot, y^{(n)} = p^{(n-1)},$$

we find that the resulting equation to be solved is similar in form, viz.,

$$f(x, p, p', \cdot \cdot \cdot, p^{(n-1)}) = 0,$$

and is of order $n - 1$. If this differential equation can be solved, we have

$$p = \phi(x, c_1, c_2, \cdot \cdot \cdot c_{n-1}),$$

and y can be obtained by the quadrature

$$y = \int \phi(x, c_1, c_2, \cdot \cdot \cdot, c_{n-1}) dx + c_n.$$

Again, if $y, y', y'', \cdot \cdot \cdot, y^{(r-1)}$ are absent, so that the equation is of the form

$$f(x, y^{(r)}, y^{(r+1)}, \cdot \cdot \cdot, y^{(n)}) = 0,$$

the transformation $y^{(r)} = v$ reduces it to one of similar form

$$f(x, v, v', \cdots, v^{(n-r)}) = 0,$$

which is of order $n - r$. If this differential equation can be solved, we have $v = \phi(x, c_1, c_2, \cdots, c_{n-r})$, and y can be obtained by r successive quadratures, i.e.,

$$y = \int\int \cdots \int \phi(x, c_1, c_2, \cdots, c_{n-r})dx^r + c_{n-r+1}x^{r-1} + c_{n-r+2}x^{r-2} + \cdots + c_{n-1}x + c_n.$$

If y and all of its derivatives except the highest are absent, the equation, when solved for the highest derivative, assumes the form

$$y^{(n)} = f(x),$$

and the solution is obtained by n successive quadratures. For, $y^{(n-1)} = \int f(x)dx + a_1$, whence $y^{(n-2)} = \int\int f(x)dx^2 + a_1x + a_2$, and so on, until we have

$$y = \int\int \cdots \int f(x)dx^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n.$$

Exercise **1.** $(1 + x^2)y'' + 1 + y'^2 = 0.$

Putting $y' = p$, we have

$$(1 + x^2)\frac{dp}{dx} + 1 + p^2 = 0, \quad \text{or} \quad \frac{dp}{1 + p^2} + \frac{dx}{1 + x^2} = 0.$$

$$\therefore \tan^{-1} p = c - \tan^{-1} x, \text{ or } p = \frac{c_1 - x}{1 + c_1x}, \text{ where } c_1 = \tan c.$$

Integrating, we have

$$c_1^2y = (c_1^2 + 1) \log (1 + c_1x) - c_1x + c_2.$$

Exercise **2.** $(xy''' - y'')^2 = y'''^2 + 1.$

Putting $y'' = v$ and solving for v, we have

$$v = xv' \pm \sqrt{v'^2 + 1}.$$

This is Clairaut's form (§IV·4) and has for solution*

$$v = y'' = cx \pm \sqrt{c^2 + 1}.$$

---

* The singular solution of this equation will also lead to a solution of the original differential equation. But this will involve only two arbitrary constants.

Integrating, we find

$$y' = \frac{c}{2}x^2 \pm x\sqrt{c^2+1} + c'.$$

Integrating again, we have the solution

$$y = \frac{c}{6}x^3 \pm \frac{x^2}{2}\sqrt{c^2+1} + c'x + c''.$$

3. $xy'' + (x^2 - 1)(y' - 1) = 0.$ **4.** $y'' = xe^x.$

5. $a^2y''^2 = (1 + y'^2)^3.$ **6.** $y'' + y'^2 + 1 = 0.$

7. $x^2y''^2 - 2xy'y''(1 + y'^2) - (1 + y'^2)^2 = 0.$

VIII·3. Independent variable absent. If x is absent, we let y' be the new dependent variable and take y as the independent one. Putting $y' = p$, we have

$$\begin{aligned} y'' &= \frac{dp}{dx} = \frac{dp}{dy}\,\frac{dy}{dx} = p\,\frac{dp}{dy}, \\ y''' &= \frac{dy''}{dx} = p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2, \\ y^{\text{iv}} &= \frac{dy'''}{dx} = p^3\frac{d^3p}{dy^3} + 4p^2\frac{dp}{dy}\,\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^3, \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

and the differential equation reduces to one of order $n - 1$. If this can be solved, we have

$$y' = \phi(y, c_1, c_2, \cdots, c_{n-1}),$$

and y can be obtained by the quadrature

$$\int\frac{dy}{\phi(y, c_1, c_2, \cdots, c_{n-1})} = x + c_n.$$

Exercise **1.** $yy'' - y'^2 - y^2y' = 0.$

Putting $y' = p$, we have $y'' = p\,\dfrac{dp}{dy}$, and the equation becomes

$$p\left(y\frac{dp}{dy} - p - y^2\right) = 0,$$

which is equivalent to two equations.

The one, $p = 0$, has the solution $y = c$, which is a solution of the original differential equation.

The other,

$$y\frac{dp}{dy} - p - y^2 = 0,$$

has the integrating factor $1/y^2$. Using this, we have

$$\frac{p}{y} = y + c.$$

Remembering that $p = dy/dx$, we have

$$\frac{dy}{y(y+c)} = dx.$$

The solution of this is readily found to be

$$\frac{y}{y+c} = ae^{cx}.$$

This is the general solution of the original differential equation. It includes the other solution $y = c$, found above, for the special value $a = \infty$. Or, if the general solution is written in the form

$$\frac{y+c}{y} = \alpha e^{-cx},$$

the special form results from putting $\alpha = 0$.

2. $yy'' + y'^2 + 1 = 0$. **3.** $2y'' = e^y$.

4. $yy'' + 2y' - y'^2 = 0$. **5.** $3y''^2 - y'y''' = 0$.

VIII·4. Equation homogeneous in *y* and its derivatives. If the differential equation

$$(1) \qquad f(x, y, y', \cdots, y^{(n)}) = 0$$

is homogeneous, of degree k, in $y, y', \cdots, y^{(n)}$ (the manner in which x appears playing no rôle), introduce the new dependent variable $v = \log y$. Then

$$y = e^v,\ y' = e^v v',\ y'' = e^v(v'' + v'^2),\ y''' = e^v(v''' + 3v''v' + v'^3), \cdots.$$

Here v appears only in the factor e^v. Substituting these values in (1), we have an equation of the form

$$e^{kv}F(x, v', v'', \cdots, v^{(n)}) = 0,$$

since (1) is homogeneous in y and its derivatives. If we neglect the non-vanishing factor e^{kv}, the resulting differential equation is seen to be free of v, the dependent variable. Hence, the method of §VIII·2 applies.

Exercise **1.** $yy'' - y'^2 - 6xy^2 = 0$.
2. $xyy'' - xy'^2 = y\sqrt{y^2 + y'^2}$.
3. $xyy'' + xy'^2 - yy' = 0$. **4.** $xyy'' + 2yy' - xy'^2 = 0$.

★VIII·5. Isobaric equation. The term isobaric, introduced in §II·4, may be extended to functions involving higher derivatives. Thus, we shall define $f(x, y, y', y'', \cdots, y^{(n)})$ as an *isobaric function of weight* r, if a number m exists such that the result of replacing x by tx, y by $t^m y$, y' by $t^{m-1}y'$, y'' by $t^{m-2}y''$, $\cdots$, $y^{(n)}$ by $t^{m-n}y^{(n)}$ is the original function multiplied by t^r; i.e., if

$$f(tx, t^m y, t^{m-1}y', \cdots, t^{m-n}y^{(n)}) \equiv t^r f(x, y, y', \cdots, y^{(n)}).$$

As before, letting $t = 1/x$, we have

$$f(x, y, y', \cdots, y^{(n)}) \equiv x^r f\left(1, \frac{y}{x^m}, \frac{y'}{x^{m-1}}, \cdots, \frac{y^{(n)}}{x^{m-n}}\right).$$

Hence, the isobaric function may be written as x^r times another function of the ratios y/x^m, y'/x^{m-1}, $\cdots$, $y^{(n)}/x^{m-n}$; thus

$$f(x, y, y', \cdots, y^{(n)}) \equiv x^r F\left(\frac{y}{x^m}, \frac{y'}{x^{m-1}}, \cdots, \frac{y^{(n)}}{x^{m-n}}\right). \tag{1}$$

Here $x, y, y', y'', \cdots, y^{(n)}$ are said to *have the weights* $1, m, m - 1, m - 2, \cdots, m - n$ respectively, and $x^a y^b y'^c \cdots y^{(n)k}$ has the weight $a + bm + c(m - 1) + \cdots + k(m - n)$.

If a polynomial in $x, y, y', \cdots, y^{(n)}$ is isobaric for some value of m, the latter may be found by selecting that value for it which causes each term of the polynomial to have the same weight. (See Ex. 1, below.) This common weight is r, the weight of the polynomial.

A differential equation, obtained by equating an isobaric function of $x, y, y', \cdots, y^{(n)}$ to zero, is said to be an *isobaric*

★ This section may be omitted in a first course.

differential equation. Dividing the equation by x^r, we see from (1) that the isobaric equation may be written in the form

$$F\left(\frac{y}{x^m}, \frac{y'}{x^{m-1}}, \cdots, \frac{y^{(n)}}{x^{m-n}}\right) = 0, \tag{2}$$

where the weight of each argument is zero. Here m may have any finite positive or negative rational value, including zero.

If $m = 0$, the differential equation (2) has the form

$$F(y, xy', x^2y'', \cdots, x^ny^{(n)}) = 0. \tag{3}$$

The Cauchy linear differential equation (§VI·11), with right-hand member zero, is an equation of this type. As in the case of a Cauchy equation, the change of the independent variable $u = \log x$ transforms any equation of the form (3) into one in which the independent variable is absent. This follows at once from the fact that this transformation results in

$$xy' = \frac{dy}{du} \equiv \mathfrak{D}y,\ x^2y'' = \mathfrak{D}(\mathfrak{D}-1)y,\ x^3y''' = \mathfrak{D}(\mathfrak{D}-1)(\mathfrak{D}-2)y,\ \cdots,$$

as was shown in §VI·11. Then the method of §VIII·3 is available.

If $m \neq 0$, make the change of dependent variable $v = y/x^m$, or $y = x^mv$. Then

$$\begin{aligned}
y' &= x^mv' + mx^{m-1}v,\\
y'' &= x^mv'' + 2mx^{m-1}v' + m(m-1)x^{m-2}v,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
y^{(k)} &= x^mv^{(k)} + kmx^{m-1}v^{(k-1)} + \frac{k(k-1)}{2!}m(m-1)x^{m-2}v^{(k-2)} + \cdots\\
&\qquad + m(m-1)\cdots(m-k+1)x^{m-k}v,\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

and

$$\frac{y}{x^m} = v,\ \frac{y'}{x^{m-1}} = xv' + mv,\ \frac{y''}{x^{m-2}} = x^2v'' + 2mxv' + m(m-1)v, \cdots,$$

$$\frac{y^{(k)}}{x^{m-k}} = x^kv^{(k)} + kmx^{k-1}v^{(k-1)} + \cdots + m(m-1)\cdots(m-k+1)v, \cdots.$$

Hence, this change of dependent variable transforms the differential equation (2) into one of form (3), for which $m = 0$. A method of solution for this case has been given above.

If the isobaric differential equation is not of weight zero [as it is in form (2)] but of weight r, its left-hand member will be that of (2) multiplied by x^r, because of (1). This extra factor is not affected by the change of dependent variable and is changed to e^{ru} by the change of independent variable.

Summing up the results obtained above, we have established the following

Theorem. *Given a differential equation which is isobaric of weight r when the weights $1, m, m-1, \cdot\cdot\cdot, m-n$ are assigned to $x, y, y', \cdot\cdot\cdot, y^{(n)}$ respectively; the result of the transformation of variables*

$$x = e^u, \quad y = ve^{mu}$$

is a differential equation from which the new independent variable is absent or may be eliminated by dividing by the non-vanishing factor e^{ru}.

The method of §VIII·3 may now be employed.

Remark. The change of variables

$$x = e^u, \quad y = vx^m$$

replaces $y^{(k)}/x^{m-k}$ by

$$(\mathfrak{D} + m)(\mathfrak{D} + m - 1) \cdot\cdot\cdot (\mathfrak{D} + m - k + 1)\, v,$$

where $\mathfrak{D}$ refers to differentiation with respect to u. When $m = 0$, we have the case considered in §VI·11.

***Exercise* 1.** $x^2yy'' - 3xyy' + x^2y'^2 + 2y^2 + x^2 = 0.$

If the weights $1, m, m-1, m-2$ are assigned to x, y, y', y'' respectively, the terms of the differential equation have the weights

$$2 + m + m - 2 \equiv 2m, \quad 1 + m + m - 1 \equiv 2m, \quad 2 + 2m - 2 \equiv 2m, \quad 2m,\ 2 \text{ respectively.}$$

Each of these is equal to 2 when $m = 1$. Hence the equation is isobaric of weight 2 when we assign to x, y, y', y'' the weights $1, 1, 0, -1$ respectively. Using our theorem, we let

$$x = e^u, \quad y = ve^u.$$

Then

$$y' = \frac{dy}{dx} = \frac{d(ve^u)}{du} \cdot \frac{du}{dx} = e^u\left(\frac{dv}{du} + v\right) \cdot e^{-u} = \frac{dv}{du} + v,$$

$$y'' = \frac{dy'}{dx} = \frac{d\left(\dfrac{dv}{du} + v\right)}{du} \cdot \frac{du}{dx} = \left(\frac{d^2v}{du^2} + \frac{dv}{du}\right)e^{-u}.$$

Putting these in the differential equation, we find, after simple reductions, that

$$e^{2u}\left[v\frac{d^2v}{du^2} + \left(\frac{dv}{du}\right)^2 + 1\right] = 0.$$

After division by e^{2u}, the differential equation is free of u. While the method of §VIII·3 may be employed, the integration in this case can be effected more readily by noting that

$$v\frac{d^2v}{du^2} + \left(\frac{dv}{du}\right)^2 = \frac{d}{du}\left(v\frac{dv}{du}\right)$$

and integrating directly. Compare §VIII·6. The rest of the exercise will be left for the student.

2. $x^3y'' + x^2y'^2 - (x^2 + 2xy)y' + x^2 + xy + y^2 = 0.$

3. $x^2yy'' + x^2y'^2 - 7xyy' - x^4 + 8y^2 = 0.$

4. $x^4yy'' + x^4y'^2 + 3x^3yy' - 1 = 0.$

VIII·6. Exact differential equations. A differential equation of order n is said to be *exact* if, when all of its terms involving the dependent variable and its derivatives are brought over to one side of the equation, that member is the derivative of some function of the independent variable, the dependent variable, and the derivatives of the latter. Equating the integrals of the two members of an exact equation leads to a new differential equation of order $n - 1$, which still remains to be solved. This new equation involves one arbitrary constant, and is known as a *first integral* of the original differential equation.

In the case of linear differential equations, a simple test for exactness can be given. For illustrative purposes, an equation of the third order will be used. Thus, consider the linear differential equation whose left-hand member is

$$P_0y''' + P_1y'' + P_2y' + P_3y. \tag{1}$$

The term P_0y''' will arise on differentiating P_0y'', for the derivative of the latter is $P_0y''' + P_0'y''$. Subtracting this exact derivative from (1), we obtain

$$(P_1 - P_0')y'' + P_2y' + P_3y, \tag{2}$$

which must evidently be exact whenever (1) is.

As before, we note that the term $(P_1 - P_0')y''$ will arise on differentiating $(P_1 - P_0')y'$, for the derivative of the latter is $(P_1 - P_0')y'' + (P_1' - P_0'')y'$. Subtracting this from (2), we obtain

$$(P_2 - P_1' + P_0'')y' + P_3y, \tag{3}$$

which must also be exact if (1) is.

The term $(P_2 - P_1' + P_0'')y'$ will arise on differentiating $(P_2 - P_1' + P_0'')y$, whose derivative is

$$(P_2 - P_1' + P_0'')y' + (P_2' - P_1'' + P_0''')y.$$

Subtracting this from (3), we obtain

$$(P_3 - P_2' + P_1'' - P_0''')y, \tag{4}$$

which must be exact if (1) is. But this can only be the case if the coefficient of y is identically zero; i.e., if

$$P_3 - P_2' + P_1'' - P_0''' \equiv 0. \tag{5}$$

Moreover, condition (5) is evidently sufficient. For, then, equation (1) results from differentiating

$$P_0y'' + (P_1 - P_0')y' + (P_2 - P_1' + P_0'')y = \int X dx + c.$$

In an analogous manner, one can show that *the necessary and sufficient condition for the exactness of the linear differential equation of the nth order*

$$P_0y^{(n)} + P_1y^{(n-1)} + \cdots + P_{n-r}y^{(r)} + \cdots + P_{n-1}y' + P_ny = X \tag{6}$$

is

$$P_n - P_{n-1}' + P_{n-2}'' - \cdots + (-1)^r P_{n-r}^{(r)} + \cdots + (-1)^n P_0^{(n)} \equiv 0. \tag{7}$$

A similar method applies to equations that are not linear, but in such cases there is no simple test for exactness; one must actually carry out the work of obtaining a first integral in order to determine whether the differential equation is exact or not. The following exercise will illustrate:

Exercise **1.** $(x + y^2)y''' + 6yy'y'' + 3y'' + 2y'^3 = 0.$

The derivative of $(x + y^2)y''$ is $(x + y^2)y''' + y'' + 2yy'y''$. Subtracting this from the left-hand member of the differential equation, we have

$$4yy'y'' + 2y'' + 2y'^3.$$

The derivative of $2yy'^2$ is $4yy'y'' + 2y'^3$. Subtracting this, we have

$$2y'',$$

which is the derivative of $2y'$. Hence a first integral is

$$(x + y^2)y'' + 2yy'^2 + 2y' = c.$$

It will be left as an exercise for the student to repeat the process and ultimately solve the differential equation.

2. $(x - 1)y'' + (x + 1)y' + y = 2x.$
3. $(x^2 + x)y'' + (3x + 2)y' + y = 4x.$
4. $(x^3 - x)y''' + (8x^2 - 3)y'' + 14xy' + 4y = 0.$
5. $(y - x)y'' + y'^2 - 2y' = 1 - \sin x.$
6. $x^3yy''' + 3x^3y'y'' + 9x^2yy'' + 9x^2y'^2 + 18xyy' + 3y^2 = 0.$
7. $(x - x^2)y'' - 3y' + 2y = 2x - 1.$
8. $x^4y''' + 6x^3y'' + (6x^2 + 1)y' = 2x + 1/x.$
9. $(x - x^2)y'' + 4y' + 2y = 0.$

VIII·7. Integrating factor. We have seen that, in general, every differential equation of the first order, when solved for y', has an integrating factor. This is not true for differential equations of higher order than the first, when solved for the highest derivative. There is a general theory of integrating factors for linear differential equations of any order, but its consideration is beyond the range of this treatise. On the other hand, there is no such theory for non-linear differential equations of higher order than the first.

In this section, we shall call attention to several types of equations for which integrating factors may be discovered by inspection or by simple tests.

1° In the case of a differential equation of the form

$$y'' = f(y), \tag{1}$$

the integrating factor $2y'$ is more or less obvious. Using it, we have

$$2y'y'' = 2f(y)y'.$$

A quadrature gives a first integral in the form

$$y'^2 = 2\int f(y)dy + c.$$

It is interesting to note that the method of §VIII·3, which applies to this differential equation because x is absent, leads to precisely the same quadrature.

This type of equation frequently occurs in problems in Dynamics. The method given here is the one often employed when a knowledge of Differential Equations is not assumed.

2° Casual inspection will enable one to note that $1/y'$ is an integrating factor for an equation of the type

$$y'' + f(x)y' + \phi(y)y'^2 = 0.\text{*} \tag{2}$$

Introducing it, we have

$$\frac{y''}{y'} + f(x) + \phi(y)y' = 0.$$

Integrating each term, we obtain

$$\log y' + \int f(x)dx + \int \phi(y)dy = c.$$

Taking exponentials, we may write this in the form

$$e^{\int \phi(y)dy}dy = c_1 e^{-\int f(x)dx}dx,$$

---

* This type of equation was first treated by Joseph Liouville (1809–1882). Let the student show that it is the differential equation corresponding to a primitive of the form $F(y) = a\Phi(x) + b$, where F and Φ are functions of their respective variables, and a and b are the arbitrary constants to be eliminated.

where the variables are separated. Integrating again, we obtain the solution

$$\int e^{\int \phi(y)dy}dy = c_1\int e^{-\int f(x)dx}dx + c_2,$$

which has the form mentioned in the footnote on the previous page.

3° It is easy to determine whether a linear differential equation, whose coefficients are polynomials, has an integrating factor of the form x^m. To do this, we multiply the equation by x^m and see whether a value can be found for m which will satisfy the condition (7) of §VIII·6. The following exercises will illustrate:

Exercise **1.** $y'' - x^2y' + xy = x$. (Ex. 1, §VII·1.)

Multiplied by x^m, this becomes

$$x^my'' - x^{m+2}y' + x^{m+1}y = x^{m+1}.$$

The test for exactness of this equation is

$$(m + 3)x^{m+1} + m(m - 1)x^{m-2} \equiv 0.$$

It is impossible to find a value of m satisfying the two equations

$$m + 3 = 0 \quad \text{and} \quad m(m - 1) = 0.$$

Hence there is no integrating factor of the form x^m for this differential equation.

Exercise **2.** $x^5y'' + (2x^4 - x)y' - (2x^3 - 1)y = 0.$

This equation is not exact, as may be verified readily.

Multiplying by x^m and applying the test for exactness, we find

$$(m^2 + 7m + 10)x^{m+3} + (m + 2)x^m \equiv 0.$$

Since $m = -2$ satisfies each of the equations

$$m^2 + 7m + 10 = 0 \quad \text{and} \quad m + 2 = 0,$$

x^{-2} is an integrating factor. Using it, we have

$$x^3y'' + (2x^2 - x^{-1})y' - (2x - x^{-2})y = 0.$$

This is exact, and the method of §VIII·6 leads to the first integral

$$x^3y' - (x^2 + x^{-1})y = c,$$

which is a linear differential equation of the first order. Its solution will be left as an exercise.

3. $x^2(1 - x^2)y'' - x^3y' - 2y = 0.$

4. $x^2y''' - 5xy'' + (4x^4 + 5)y' - 8x^3y = 0.$

5. $y'' + 2 \cot x \cdot y' + 2 \tan y \cdot y'^2 = 0.$

6. $xyy'' + xy'^2 - yy' = 0.$ (Ex. 3, §VIII·4.) [Solve for y''.]

7. $y'' + y = \sec x.$ (Ex. 1, §VI·9.) [$\cos x$ is an integrating factor.]

VIII·8. Linear equation with particular integral known. If the equation is linear and of any order, and if a particular integral is known when the right-hand member is made zero, the method 1°, §VII·1, applies.* Thus, let the equation be

$$P_0y^{(n)} + P_1y^{(n-1)} + \cdots + P_{n-1}y' + P_ny = X,$$

and let y_1 be the known particular integral. Putting $y = y_1v$, we have

$$\begin{aligned} y' &= y_1v' + y_1'v, \\ y'' &= y_1v'' + 2y_1'v' + y_1''v, \\ &\cdots\cdots\cdots \\ y^{(n)} &= y_1v^{(n)} + \cdots + y_1^{(n)}v. \end{aligned}$$

Making this substitution, we have

$$P_0y_1v^{(n)} + \cdots + [P_0y_1^{(n)} + P_1y_1^{(n-1)} + \cdots + P_ny_1]v = X.$$

By hypothesis, the coefficient of v is zero. Hence the equation reduces to one of order $n - 1$ when v' is taken as the dependent variable.

Exercise **1.** $(x^2 - 2x + 2)y''' - x^2y'' + 2xy' - 2y = 0.$

Here $y = x$ is a particular solution. Putting $y = xv$, we have

$$(x^3 - 2x^2 + 2x)v''' - (x^3 - 3x^2 + 6x - 6)v'' = 0.$$

Since the coefficient of v' is also zero, the equation may be written in the form

$$\frac{dv''}{v''} = \frac{x^3 - 3x^2 + 6x - 6}{x^3 - 2x^2 + 2x}\,dx = dx - \frac{3dx}{x} + \frac{2x - 2}{x^2 - 2x + 2}\,dx.$$

---

* The hint given there, as to how a particular integral may at times be found, applies equally well here.

Hence,

$$\log v'' = x - 3\log x + \log(x^2 - 2x + 2) + c,$$

and

$$v'' = \frac{d^2v}{dx^2} = \frac{c_1(x^2 - 2x + 2)e^x}{x^3}.$$

Integrating successively, we find

$$\frac{dv}{dx} = \frac{c_1(x-1)e^x}{x^2} + c_2,$$

and

$$v = \frac{c_1e^x}{x} + c_2x + c_3.$$

Therefore,

$$y = c_1e^x + c_2x^2 + c_3x.$$

***Exercise* 2.** $xy''' - y'' - xy' + y = 1 - x^2$.

By inspection, it is seen that e^x, e^{-x}, x are particular integrals; hence we know at once that the complementary function is

$$Y = c_1e^x + c_2e^{-x} + c_3x.$$

The student should verify that, by the method of variation of parameters (§VI·9), this becomes the general solution when c_1, c_2, c_3 are replaced by $\frac{e^{-x}}{2}(x + 2 - x^{-1}) + k_1$, $\frac{e^x}{2}(x^{-1} + 2 - x) + k_2$, $x + \frac{1}{x} + k_3$ respectively. Hence the solution is

$$y = k_1e^x + k_2e^{-x} + k_3x + x^2 + 3.$$

In order to secure practice in the general method of this section, let the student solve this equation by that method. It is suggested that he take $y_1 = e^{-x}$, and then use the hint given in the last part of the footnote in §VII·1, 1° to find a particular solution for the resulting differential equation of the second order, when the right-hand member is made zero.

3. $x^3y''' - (2x^3 + 3x^2)y'' + (x^3 + 4x^2 + 6x)y' - (x^2 + 4x + 6)y = 0.$

In this section, we have brought out the fact that if a particular integral is known for a linear differential equation, when the right-hand member is made zero, a transformation of the dependent variable can be found which reduces the order of the equation by unity; so that the problem of solving the linear differential equation is reduced to that of solving one of order $n - 1$, with an additional quadrature.*

Moreover, *if r linearly independent particular integrals are known for a linear differential equation of order n, when the right-hand member is made zero, the problem of its solution can be reduced to that of solving one of order $n - r$ and r quadratures.*

Let the complementary function of the differential equation be

$$y = c_1y_1 + c_2y_2 + c_3y_3 + \cdots + c_ny_n.$$

If y_1 is known, the transformation $y = y_1v$, or $v = y/y_1$, changes the linear differential equation into one in which v is the dependent variable, and its complementary function is

$$v = c_1 + c_2y_2/y_1 + c_3y_3/y_1 + \cdots + c_ny_n/y_1. \tag{1}$$

Because of the term c_1, we note again that the new equation does not contain a term in v.

If, now, y_2 is also known, it follows from (1) that y_2/y_1 is a known particular integral for the differential equation having v for dependent variable, when the right-hand member is made zero. Hence $\dfrac{d(y_2/y_1)}{dx}$ is a particular integral for the differential equation of order $n - 1$ in v'. So the additional transformation $v' = u\,\dfrac{d(y_2/y_1)}{dx}$ results in a linear differential equation of order $n - 1$ in u with the term in u absent; i.e., in one of order $n - 2$ in u'.

By induction, we can establish the theorem, when r particular integrals are known.

To illustrate the steps involved here, the student should apply them to the exercises of this section. Care in selecting the order, in which the particular integrals are taken, will simplify the process.

---

* As illustrated in Ex. 1, it is possible that the order may be reduced by more than one. If the order is reduced by k, the number of quadratures is also k.

VIII·9. Transformation of variables. In §§VIII·2, VIII·3, VIII·4, VIII·5, VIII·8 we noted that a change of variables transforms the given differential equation into one which is more readily solvable. Other cases arise occasionally when a transformation of variables will aid materially in obtaining a solution. No general rule for this can be formulated. The form of the equation will frequently suggest the transformation to be tried.

***Exercise* 1.** $x^2yy'' + (xy' - y)^2 = 0.$

The set of terms $(xy' - y)^2$ suggests the transformation $y = vx$. When this transformation is made, the equation reduces to

$$xvv'' + 2vv' + xv'^2 = 0.$$

This is exact, and has for first integral

$$xvv' + \tfrac{1}{2}v^2 = c.$$

This also is exact; whence

$$xv^2 = c_1x + c_2,$$

or

$$y^2 = c_1x^2 + c_2x.$$

Arrangement of the terms in the following way

$$x^2(yy'' + y'^2) - 2xyy' + y^2 = 0$$

suggests the transformation $y^2 = v$. Let the student solve the problem by making this transformation.

2. $x^3y'' - (xy' - y)^2 = 0.$

3. $yy'' - y'^2 = y^2 \log y - x^2y^2.$ [Let $\log y = v$, i.e., $y = e^v$.]

4. $\sin^2 x \cdot y'' - 2y = 0.$ [Let $\cot x = u$.]

If the more obvious transformation $\sin x = u$ is made, the resulting equation can be made exact by multiplying by a proper power of u, and can then be integrated.

5. $x^2(2yy'' - y'^2) - 4xyy' + 8y^2 = 0.$ (Compare Ex. 8, §VIII·10.)

This equation is homogeneous in y and its derivatives, thus suggesting the transformation $\log y = v$. This does transform the equation to one reducible to the first order; but it is a Riccati equation (§IX·11), for which a method of solution is not obvious.

Further examination of the given equation causes one to note that the combination $2yy'' - y'^2$ suggests the second derivative of $y^{1/2}$. Division by $4y^{3/2}$ reduces the coefficient of x^2 to this second derivative. The remaining terms are then closely associated with $y^{1/2}$. Hence the transformation $y = v^2$ should be tried.

6. $x(yy'' + y'^2) + 3yy' = 2x^3$.

The combinations $yy'' + y'^2$ and yy' suggest the derivatives of y^2 and, therefore, the transformation $y^2 = v$.

VIII·10. Summary. The number of classes of differential equations of higher order than the first for which a general method of solution is known is very small.

1° If the dependent variable is absent, let the lowest derivative* that appears be the new dependent variable (§VIII·2).

2° If the independent variable is absent, let the first derivative of the dependent variable be the new dependent variable, and take the old dependent variable as the new independent one (§VIII·3).

3° If the equation is linear and a particular integral y_1 can be found when the right-hand member is made zero, let $y = y_1v$, and in the transformed equation let v' be the new dependent variable. If r linearly independent particular integrals can be found, the problem can be reduced to that of solving an equation of order $n - r$ and quadratures (§VIII·8).

4° If the equation is linear and of the second order, the methods of Chapter VII may apply (§VII·3).

5° If the equation is homogeneous in y and its derivatives, let $\log y = v$ (§VIII·4).

6° If the equation becomes isobaric when the weights $1, m, m - 1, \cdot \cdot \cdot, m - n$ are assigned to $x, y, y', \cdot \cdot \cdot, y^{(n)}$ respectively, let $x = e^u$, $y = vx^m = ve^{mu}$ (§VIII·5).

If none of the above cases occur, test the equation for exactness (§VIII·6).

If the equation is not exact, some special device may be tried, such as finding an integrating factor (§VIII·7), or finding some suitable transformation (§VIII·9).

---

* If this is also the highest derivative that appears, let the next lower derivative be the new variable.

As a final resort, the method of integrating in series may be tried (Chap. IX).

Exercise **1.** $y'' = y'^2 + 1$.

2. $(1 - x^2)y'' - xy' = 2$. **3.** $y'' + yy' = 0$.

4. $(1 + x^3)y''' + 9x^2y'' + 18xy' + 6y = 0$.

5. $(x^2 - x)y'' + (4x + 2)y' + 2y = 0$.

6. $y(1 - \log y)y'' + (1 + \log y)y'^2 = 0$. **7.** $y'' = e^{2y}$.

8. $x^2yy'' - x^2y'^2 - 4xyy' + 8y^2 = 0$.

9. $xy'' - (2x^2 + 1)y' + 4x^3 = 0$.

10. $(1 + y'^2)y''' - 3y'y''^2 = 0$.

11. $3y'y''y^{\text{iv}} - 4y'y'''^2 - 3y''^2y''' = 0$.

12. $xyy'' - yy' - xy'^2 + (1 - 2x)y^2 = 0$.

13. $x(x + 2y)y'' + 2xy'^2 + 4(x + y)y' + 2y + x^2 = 0$.

14. $(x - x^3)y'' - y' + x^3 = 0$.

15. $x^4y'' + (2x^3 + x^2)y' + 1 = 0$.

16. $\sin x \cdot y'' - \cos x \cdot y' + 2 \sin x \cdot y = 0$.

17. Determine the curves for which the radius of curvature at any point is equal in length to the part of the normal between that point and the x-axis, (a) when the two have the same direction, (b) when they have opposite directions.

[The radius of curvature $= (1 + y'^2)^{3/2}/y''$. The part of the normal, involved here, extends from the point on the curve to the x-axis. When the radius of curvature is drawn towards the x-axis also, y and y'' have opposite signs; when it is drawn away from the x-axis, y and y'' have the same sign.]

18. Determine the curves for which the radius of curvature at any point is equal in length to twice the part of the normal between that point and the x-axis, (a) when the two have the same direction, (b) when they have opposite directions.

19. Determine the curves for which the radius of curvature at any point is equal in length to k times the cube of the part of the normal between that point and the x-axis.

20. A particle which starts from a point on the axis of x, at a distance a from the origin, moves uniformly in a direction parallel to the axis of y. It is pursued by a particle which starts at the same time from the origin, and travels with a velocity which is n times that of the former. Find the path of the latter.

[A path of this kind is usually referred to as a *curve of pursuit*. In the general case, its differential equation may be obtained from the following considerations: Let (x, y) be the coördinates of the pursuing point, (ξ, η) those of the point pursued. The path of the latter being known, we have given

$$f(\xi, \eta) = 0. \tag{1}$$

Since the point pursued is always in the tangent to the curve of pursuit, we have

$$\eta - y = y'(\xi - x). \tag{2}$$

The equations (1) and (2) determine ξ and η in terms of x, y, y'. If the velocities of the point pursued and pursuing point are in the ratio $1 : n$, we have

$$n\sqrt{d\xi^2 + d\eta^2} = \sqrt{dx^2 + dy^2};$$

or, taking x as the independent variable, and squaring, we have

$$n^2(\xi'^2 + \eta'^2) = 1 + y'^2. \tag{3}$$

Substituting in this the values of ξ' and η' obtained from (1) and (2), we find the differential equation of the curve of pursuit.]

21. Find the velocity of the weighted end of a simple pendulum of length l, swinging in a vacuum, if at the time $t = 0$, $v = 0$ and $\theta = \alpha$, where α is too large to permit $\sin \alpha$ to be replaced by α as a first approximation. (See §VI·13, 1°.)

22. A particle moves in a straight line attracted by a force, centered at a point on the line and varying inversely as the square of the distance between it and the moving particle. [Equation of motion is $d^2x/dt^2 = -k^2/x^2$.] If it starts with zero velocity at a distance a from the center of the force,

(*a*) find its velocity at any point in its path,

(*b*) find the time required to reach that point,

(*c*) find how far it will have to move in order to acquire the same velocity as that possessed by another particle when it reaches the point a after having started at infinity with zero velocity and moving in accordance with the same law,

(d) find the velocity with which a body (a meteorite, for example) will strike the surface of the earth if it falls from a distance h above the surface.

[Gravity acts according to the inverse square law. Acceleration due to it, at the earth's surface, is usually designated by $-g$. Hence $k^2 = gR^2$, if R is the radius of the earth. Here $a = R + h$.]

CHAPTER IX

INTEGRATION IN SERIES

IX·1. Existence theorems. The number of classes of differential equations that can be integrated by quadratures or other purely elementary means is very small, compared with the number of possible classes of equations. The question arises whether every differential equation can have a solution.

The existence of solutions for extensive classes of differential equations is established in the general theory of the subject. But a proper understanding of the proof of the theorems establishing them requires a knowledge of the Theory of Functions, which is not assumed here.

1° The existence theorem for the differential equation of the first order, $dy/dx = F(x, y)$, may be stated as follows:

If $F(x, y)$ can be developed by Taylor's theorem as a power series in $x - x_0$ and $y - y_0$, absolutely convergent when the absolute values of these elements do not exceed certain definite amounts, there exists a unique solution, $y = y(x)$, which satisfies the initial condition $y_0 = y(x_0)$.*

In some proofs of the theorem, y is found in the form of an infinite series

$$y_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_n(x - x_0)^n + \cdots,$$

which satisfies the equation when substituted in it for y, and is convergent for values of x sufficiently near to x_0.

Since y_0 may be chosen arbitrarily (within certain limits, however), we see that in the case of a differential equation of the first order *one* arbitrary constant enters.

---

* For example, $F(x, y)$ cannot be so developed, if it becomes infinite or indeterminate when $x = x_0$, $y = y_0$, or if it is not single-valued in the region of x_0 or of y_0. By the *region* of x_0 we mean the aggregate of those values of x which differ from x_0, in absolute value, by less than some definite, positive number.

Remark. The existence theorem gives a sufficient condition for a unique solution, and, moreover, it suggests a possible form for that solution. Failure to satisfy the condition of the existence theorem is usually due to the choice of values for x_0 and y_0. But an equation may have a solution, even for values of these quantities for which the condition of the existence theorem is not satisfied. In general, but not necessarily always, such solutions will not be developable by Taylor's theorem in powers of $x - x_0$, or they will not be unique. A few simple examples will illustrate:

Thus, $\dfrac{dy}{dx} = \dfrac{y}{x}$, where $\dfrac{y}{x}$ becomes indeterminate when $x = y = 0$, and infinite when $x = 0$, $y \neq 0$, has the solution $y = cx$. Here y takes the value 0 when $x = 0$. It is expressed as a (finite) Taylor series, but it will be noted that c is undetermined by the condition that $y = 0$ when $x = 0$; that is, unlike the cases coming under the existence theorem, there is an indefinite number of solutions satisfying the initial condition. Moreover, it should be noted that it is impossible to find a finite value for c that will enable us to assign a value to y other than 0 when $x = 0$.

Again, $\dfrac{dy}{dx} = \dfrac{x+y}{x}$, where $\dfrac{x+y}{x}$ becomes indeterminate when $x = 0$, $y = 0$, and infinite when $x = 0$, $y \neq 0$, has the solution $y = x \log x + cx$. Here y takes the value 0 when $x = 0$. But it is not possible to express the solution in the form of a Taylor series in powers of x. In this case, also, we have an indefinite number of solutions for the one initial value 0, and no solution for any other initial value of y when $x = 0$.

The differential equation $2\dfrac{dy}{dx} = \dfrac{1+2x}{y}$ has the general solution $y = \sqrt{c + x + x^2}$. Here $\dfrac{1+2x}{y} = \infty$, when $x = x_0$, $y = 0$. In order that $y = 0$ when $x = x_0$, we must have $c = -x_0 - x_0^2$. We have then the single solution $y = \sqrt{x - x_0 + x^2 - x_0^2}$. This is not developable by Taylor's theorem in powers of $x - x_0$, except when $x_0 = -\frac{1}{2}$.

The differential equation $\frac{dy}{dx} = \sqrt{y}$ has the general solution $4y = (x + c)^2$. Here $\sqrt{y}$ is not single-valued in any region surrounding $y = 0$. Corresponding to the initial condition $x = x_0$, $y = 0$, we have the two solutions $4y = (x - x_0)^2$ and $y = 0$.

2° Before presenting the existence theorem for a differential equation of the nth order

$$\frac{d^n y}{dx^n} = F\left(x, y, \frac{dy}{dx}, \cdot \cdot \cdot, \frac{d^{n-1}y}{dx^{n-1}}\right),$$

we shall replace the differential equation by a system of n differential equations of the first order. This is done by the introduction of $(n - 1)$ new variables

$$y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2}, \cdot \cdot \cdot, y_{n-1} = \frac{d^{n-1}y}{dx^{n-1}},$$

thus giving the system

$$\begin{aligned} \frac{dy}{dx} &= y_1, \\ \frac{dy_1}{dx} &= y_2, \\ &\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \frac{dy_{n-1}}{dx} &= F(x, y, y_1, \cdot \cdot \cdot, y_{n-1}). \end{aligned}$$

This is a special case of the more general system

$$\begin{aligned} \frac{dy}{dx} &= f_1(x, y, z, \cdot \cdot \cdot, w), \\ \frac{dz}{dx} &= f_2(x, y, z, \cdot \cdot \cdot, w), \\ &\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \frac{dw}{dx} &= f_n(x, y, z, \cdot \cdot \cdot, w), \end{aligned}$$

where $y, z, \cdot \cdot \cdot, w$ are n functions of x.

The existence theorem for this system of differential equations may be stated as follows:

If $f_1, f_2, \cdots, f_n$ *are regular* in the regions of* $x_0, y_0, z_0, \cdots, w_0$, *a single set of functions* $y, z, \cdots, w$ *exists, satisfying the system of equations and taking the values* $y_0, z_0, \cdots, w_0$ *respectively when* x *takes the value* x_0.

In some proofs of the theorem these functions are found in the form of infinite series

$$\begin{aligned} y &= y_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots, \\ z &= z_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots + b_n(x - x_0)^n + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w &= w_0 + k_1(x - x_0) + k_2(x - x_0)^2 + \cdots + k_n(x - x_0)^n + \cdots, \end{aligned}$$

which are convergent when x is sufficiently close to x_0.

Here $y_0, z_0, \cdots, w_0$ may be chosen arbitrarily (within the limits stated in the previous footnote). Thus, the general solution of a system of n equations of the first order with n dependent variables involves n arbitrary constants. We noted above that an equation of the nth order, with one dependent variable, may be replaced by a system of n equations of the first order. Hence we see how we can obtain the following existence theorem for a differential equation of the nth order, as a consequence of the one, given above, for a system of n differential equations of the first order involving n dependent variables:

If $F(x, y, y', \cdots, y^{(n-1)})$ *is regular in the regions of* $x_0, y_0, y_0', \cdots, y_0^{(n-1)}$, *there exists a unique solution,* $y = y(x)$, *for the differential equation*

$$y^{(n)} = F(x, y, y', \cdots, y^{(n-1)}),$$

satisfying the conditions

$$y(x_0) = y_0, y'(x_0) = y'_0, \cdots, y^{(n-1)}(x_0) = y_0^{(n-1)}.$$

If $y(x)$ is expressed as the power series

$$y(x) = c_0 + c_1(x - x_0) + \cdots + c_{n-1}(x - x_0)^{n-1} + c_n(x - x_0)^n + \cdots,$$

---

* For our purposes it will be sufficient to state that a function of $x, y, z, \cdots, w$ is *regular* in the regions of $x_0, y_0, z_0, \cdots, w_0$ if it can be developed, by Taylor's theorem, as a power series in $x - x_0$, $y - y_0$, $z - z_0, \cdots, w - w_0$, absolutely convergent when these elements do not exceed certain definite amounts in absolute value.

the choice of $y_0, y_0', \cdots, y_0^{(n-1)}$, determining the unique solution, is precisely that of the first n constants in the series. Since this choice is arbitrary (within certain limits), we see that *the solution involves n arbitrary constants.*

Geometrically, this means:

Of the single infinity of integral curves of a differential equation of the first order and degree, a single curve passes through a given point, in general; i.e., where the conditions of the existence theorem are satisfied.

Of the double infinity of integral curves of a differential equation of the second order and first degree, a single curve passes through a given point, in a given direction, in general.

Of the triple infinity of integral curves of a differential equation of the third order and first degree, a single curve passes through a given point, in a given direction, having a given curvature at that point, in general.

IX·2. Singular solution. In the existence theorem of the previous section stress should be laid upon the fact that the existence of a unique solution of $y' = F(x, y)$, taking the value y_0 when $x = x_0$, is assured only when $F(x, y)$ is regular in the regions of x_0 and y_0. When x and y are real variables, we shall speak of the region of (x_0, y_0).

If the differential equation is given in the form

$$f(x, y, y') = 0,$$

we know, from an existence theorem for implicit functions,* that y' is expressible as a function of x and y regular in the region of (x_0, y_0), provided

$$\frac{\partial f(x, y, y')}{\partial y'} \neq 0$$

in this region. But, if

$$\frac{\partial f}{\partial y'} = 0,$$

---

* A proof of this theorem will be found in many works on Analysis, under the head of Implicit Functions.

the expression for y' in terms of x and y ceases to be single-valued. In this case, the existence theorem of 1°, §IX·1 does not assert the existence of a unique solution. As a matter of fact, a solution does not exist there, in general. For, solving $f = 0$ and $\partial f/\partial y' = 0$ for y and y', we obtain

$$y = \phi(x), y' = \phi_1(x);$$

and only in exceptional cases will

$$\phi_1(x) = \frac{d\phi(x)}{dx}.$$

If it should happen that $\phi_1(x) = d\phi(x)/dx$, then $y = \phi(x)$ is a solution of the equation; and, since it is usually distinct from the general solution, it is a singular solution. Moreover, it is identical with the singular solution studied in Chapter V.

Similarly, in the case of differential equations of higher order than the first, singular solutions may occur. Thus, if there is a solution of $f(x, y, y', \cdot\cdot\cdot, y^{(n)}) = 0$ for which $\partial f/\partial y^{(n)}$ also vanishes, this solution is, in general, a singular one.

IX·3. Solution in series. The following method for finding the solution when the conditions of the existence theorem are satisfied, is suggested by the fact that a solution exists in the form of an infinite series:

The dependent variable is assumed to be equal to a power series with undetermined coefficients. The latter are determined by substituting this value of the dependent variable and the values of its derivatives, obtained by differentiating* the series, in the differential equation, and then equating coefficients of the various powers of the independent variable.

Frequently, no general law for the coefficients in the solution is apparent. Even in such cases it is possible, at times, to determine the number of coefficients necessary to obtain a result accurate to within a given error, corresponding to a given set of initial or boundary conditions.

---

* The derivative of a function, defined by a power series in a given interval, is represented, inside the same interval, by the series of the derivatives of the terms of the original series.

At other times a general law for the coefficients can be found. This is often true in the case of linear differential equations.

The following exercises will illustrate. We shall simplify the work by assuming that $x_0 = 0$ (which is equivalent to a preliminary change of variable $x - x_0 = \bar{x}$).

***Exercise* 1.** $y' = x + y^2$.

Here $x + y^2$ is regular in the region of any finite point of the plane. Put

$$y = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots.$$

Substituting, we must have

$$c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1} + \cdots \equiv x + (c_0 + c_1x + \cdots)^2.$$

Equating coefficients, we have

$$\begin{aligned} c_1 &= c_0^2 & \therefore c_1 &= c_0^2, \\ 2c_2 &= 2c_0c_1 + 1 & \therefore c_2 &= \tfrac{1}{2} + c_0^3, \\ 3c_3 &= 2c_0c_2 + c_1^2 & \therefore c_3 &= \tfrac{1}{3}c_0 + c_0^4, \\ 4c_4 &= 2c_0c_3 + 2c_1c_2 & \therefore c_4 &= \tfrac{5}{12}c_0^2 + c_0^5, \\ & \cdots\cdots\cdots\cdots\cdots \\ 2kc_{2k} &= 2c_0c_{2k-1} + 2c_1c_{2k-2} + \cdots + 2c_{k-1}c_k, \\ (2k+1)c_{2k+1} &= 2c_0c_{2k} + 2c_1c_{2k-1} + \cdots + 2c_{k-1}c_{k+1} + c_k^2, \\ & \cdots\cdots\cdots\cdots\cdots\cdots. \end{aligned}$$

Here each coefficient can be calculated in terms of the preceding ones, and, consequently, in terms of the single one c_0, which is the value assumed by y when $x = 0$. We can calculate as many as we wish, but no general law is apparent. Our solution has the form

$$y = c_0 + c_0^2x + (\tfrac{1}{2} + c_0^3)x^2 + (\tfrac{1}{3}c_0 + c_0^4)x^3 + \cdots.$$

For certain values of c_0, it is possible to determine the number of terms necessary to obtain a value for y accurate to within a given error.

If, for example, $|c_0| < 1$, an upper bound for each of the other coefficients can be named. Thus, if $|c_0| \overline{<} \frac{1}{2}$, each of the coefficients is less than 1, and the series is term for term less in absolute value than the geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots,$$

which is convergent for $-1 < x < 1$. The error involved in using r terms of the series for the value of y, for a given value of x in the interval mentioned above, is less in absolute value than $\dfrac{|x|^r}{1-|x|}$.

Exercise **2.** $y'' + x^2y' + xy = 0$.

Putting

$$y = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots,$$

we find

$$\begin{aligned} y'' &= 2c_2 + 2\cdot 3c_3x + 3\cdot 4c_4x^2 + \cdots + (n+1)(n+2)c_{n+2}x^n + \cdots, \\ x^2y' &= c_1x^2 + \cdots + (n-1)c_{n-1}x^n + \cdots, \\ xy &= c_0x + c_1x^2 + \cdots + c_{n-1}x^n + \cdots. \end{aligned}$$

Taking the sum of these, and equating the coefficient of each power of x to zero, we have

$$c_2 = 0,\ c_3 = -\frac{c_0}{2\cdot 3},\ c_4 = -\frac{2c_1}{3\cdot 4},\ \cdots,\ c_{n+2} = -\frac{nc_{n-1}}{(n+1)(n+2)},\ \cdots.$$

From this law for the coefficients we can write the general solution in the form

$$\begin{aligned} y = c_0\Big[1 - \frac{x^3}{3!} + \frac{4^2x^6}{6!} - \frac{4^2\cdot 7^2x^9}{9!} + \cdots \\ + (-1)^r\frac{4^2\cdot 7^2\cdot 10^2\cdots(3r-2)^2x^{3r}}{(3r)!} + \cdots\Big] \\ + c_1\Big[x - \frac{2^2x^4}{4!} + \frac{2^2\cdot 5^2x^7}{7!} - \frac{2^2\cdot 5^2\cdot 8^2x^{10}}{10!} + \cdots \\ + (-1)^r\frac{2^2\cdot 5^2\cdot 8^2\cdots(3r-1)^2x^{3r+1}}{(3r+1)!} + \cdots\Big]. \end{aligned}$$

Here c_0 and c_1 are the values assumed by y and y' respectively, when $x = 0$.

3. $y'' = xy$. **4.** $y'' = xy + y^2$.

5. $y'' - x^2y' + xy = x$. (Ex. 1, §VII·1.)

6. $(1 - x^2)y'' + 2xy' + 4y = 0$.

IX·4. Existence theorem for linear differential equations. While the existence theorems mentioned in §IX·1 apply to both linear and non-linear differential equations, there are others which apply to linear differential equations only. We shall call attention to one here, which will have a bearing on the simple types of equations considered in the remainder of this chapter. We shall restrict our considerations to linear differential equations of the second order, although their extension to equations of higher orders involves no new principles.

Consider the linear differential equation

$$X_0y'' + X_1y' + X_2y = 0. \tag{1}$$

In order to simplify the statement of our existence theorem, we shall divide through by X_0 and write the equation in the form

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

If a is not a singular point* of either $P(x)$ or $Q(x)$, the existence theorem of 2°, §IX·1 applies. Here a is the x_0 of the theorem.

For this case, as well as for the one in which a is a singular point of $P(x)$ or $Q(x)$, but of the special type mentioned in the theorem, we have the following existence theorem, which may be stated in two parts. Part (*a*) applies to solutions expressed in terms of series convergent in the regions of finite points of the plane; part (*b*) applies to solutions holding in the region of infinity.

(*a*) *If* $(x - a)P(x)$ *and* $(x - a)^2Q(x)$ *can be developed by Taylor's theorem in integral powers of* $(x - a)$, *there exist two independent particular integrals (giving the general solution) of the form*

$$(x-a)^{m_1}P_1(x-a) \text{ and } (x-a)^{m_2}[P_2(x-a)+\alpha P_1(x-a)\log(x-a)],$$

where $P_1(x - a)$ *and* $P_2(x - a)$ *are power series in* $(x - a)$, *convergent in any circle*† *having its center at* a *and not including any other singular point of* P *or* Q *except* a. *The constant* α *equals*

---

* If a function is not regular in the region of a, the latter is said to be a *singular point* of the function.

† In this theory, the variables are taken to be complex quantities. The intervals of convergence for real values of x are determined by the circles of convergence when x is a complex variable.

zero if $m_1 - m_2$ is not an integer or zero. The constant α may be zero even if $m_1 - m_2$ is an integer different from zero. But the theorem gives the general case.

(*b*) *If* $xP(x)$ and $x^2Q(x)$ *can be developed in power series in* $1/x$, *convergent for sufficiently large values of* x, *there exist two independent particular integrals of the form*

$$x^{m_1}P_1\left(\frac{1}{x}\right) \quad \text{and} \quad x^{m_2}\left[P_2\left(\frac{1}{x}\right) + \alpha P_1\left(\frac{1}{x}\right)\log\left(\frac{1}{x}\right)\right],$$

where $P_1(1/x)$ *and* $P_2(1/x)$ *are power series in* $1/x$, *convergent for values of* x, *larger in absolute value than the greatest of the finite singular points of* P *and* Q. *The constant* α *equals zero if* $m_1 - m_2$ *is not an integer or zero.* Here, also, α may be zero even if $m_1 - m_2$ is an integer different from zero.

This theory of linear differential equations is usually known as the Fuchs theory after L. Fuchs, whose epoch-making memoir, which appeared in 1866 in Vol. 66 of *Crelle's Journal für die reine und angewandte Mathematik,* was the first of a series developing his theory.

IX·5. Method* of solution for a linear differential equation. In the general case, the calculation of the coefficients in the power series arising in the forms suggested by the existence theorem of the previous section is usually difficult. We shall confine ourselves to the special case of a linear differential equation whose coefficients are polynomials in the independent variable, and where the result of putting $y = x^m$ in the left-hand member of the differential equation is expressible in terms of *two* distinct powers of x.

In the case of the Cauchy equation (§VI·11), there results a single power of x.†

---

* This method is due to G. Frobenius, whose memoir, presenting the method, appeared in *Crelle's Journal*, Vol. 76 (1873).

† Thus, putting $y = x^m$ in the left-hand member of (1), §VI·11, we have

$$[k_0m(m-1)\cdots(m-n+1) + k_1m(m-1)\cdots(m-n+2) + \cdots + k_{n-1}m + k_n]x^m.$$

Equating the coefficient of x^m to zero and solving for m, we obtain, in general, n distinct particular solutions and, therefore, the complementary function. It is readily seen to be the same as that found in §VI·11.

Before presenting the method of this section in a formal manner, we shall illustrate it, in part, by solving the equation

$$2x^2y'' - xy' + (1 - 2x^2)y = 0.$$

Putting $y = x^m$ in the left-hand member, we obtain

$$(2m - 1)(m - 1)x^m - 2x^{m+2}.$$

Here the two distinct powers of x differ by 2, and m is the smaller exponent. Hence, if we put

(1) $$y = c_0x^m + c_1x^{m+2} + c_2x^{m+4} + \cdots + c_rx^{m+2r} + \cdots,$$

we shall have a solution if

1° m is so chosen that $(2m - 1)(m - 1) = 0$, i.e., $m = 1$ or $\frac{1}{2}$;

2° the c's are so chosen that the remaining terms arising on putting (1) in the left-hand member of the differential equation cancel each other in pairs; i.e.,

(2) $$\begin{cases} (2m + 3)(m + 1)c_1 - 2c_0 = 0, \\ (2m + 7)(m + 3)c_2 - 2c_1 = 0, \\ (2m + 11)(m + 5)c_3 - 2c_2 = 0, \\ \cdots\cdots\cdots\cdots\cdots \\ (2m + 4r - 1)(m + 2r - 1)c_r - 2c_{r-1} = 0, \\ \cdots\cdots\cdots\cdots\cdots \end{cases}$$

For $m = 1$, we have

$$\begin{aligned} 2\cdot5c_1 &= 2c_0 & \therefore c_1 &= \frac{c_0}{1\cdot5}, \\ 4\cdot9c_2 &= 2c_1 & \therefore c_2 &= \frac{c_1}{2\cdot9} = \frac{c_0}{1\cdot2\cdot5\cdot9}, \\ 6\cdot13c_3 &= 2c_2 & \therefore c_3 &= \frac{c_2}{3\cdot13} = \frac{c_0}{1\cdot2\cdot3\cdot5\cdot9\cdot13}, \\ &\cdots\cdots\cdots \\ 2r(4r+1)c_r &= 2c_{r-1} & \therefore c_r &= \frac{c_{r-1}}{r(4r+1)} = \frac{c_0}{r!\,5\cdot9\cdot13\cdots(4r+1)}, \\ &\cdots\cdots\cdots \end{aligned}$$

Hence

$$c_0y_1 \equiv c_0\left[x + \frac{x^3}{1\cdot5} + \frac{x^5}{2!\,5\cdot9} + \cdots + \frac{x^{2r+1}}{r!\,5\cdot9\cdot13\cdots(4r+1)} + \cdots\right]$$

is a particular integral.

For $m = \frac{1}{2}$, we have

$$\begin{array}{ll}
\frac{3}{2}\cdot 4c_1 = 2c_0 & \therefore c_1 = \frac{c_0}{1\cdot 3}, \\
\frac{7}{2}\cdot 8c_2 = 2c_1 & \therefore c_2 = \frac{c_1}{2\cdot 7} = \frac{c_0}{1\cdot 2\cdot 3\cdot 7}, \\
\frac{11}{2}\cdot 12c_3 = 2c_2 & \therefore c_3 = \frac{c_2}{3\cdot 11} = \frac{c_0}{3!\,3\cdot 7\cdot 11}, \\
\cdots\cdots & \cdots\cdots\cdots\cdots \\
\frac{4r-1}{2}\cdot 4rc_r = 2c_{r-1} & \therefore c_r = \frac{c_{r-1}}{r(4r-1)} = \frac{c_0}{r!\,3\cdot 7\cdot 11\cdots(4r-1)}, \\
\cdots\cdots & \cdots\cdots\cdots\cdots
\end{array}$$

Hence

$$c_0y_2 \equiv c_0x^{1/2}\left[1 + \frac{x^2}{1\cdot 3} + \frac{x^4}{2!\,3\cdot 7} + \cdots + \frac{x^{2r}}{r!\,3\cdot 7\cdot 11\cdots(4r-1)} + \cdots\right]$$

is a second particular integral, independent of the other, and the general solution is

$$y = Ay_1 + By_2.$$

By direct examination of the series defining y_1 and y_2, it is seen that these are convergent for all finite values of x. This is also assured by part (a) of the existence theorem of the previous section, since $x = 0$ is the only finite singular point of the coefficients of the differential equation when the leading coefficient is made unity.

We shall now present the method of this section, which applies to some very important linear differential equations of the second order.

Suppose that the result of substituting* x^m for y in the left-hand member of the differential equation (1) of §IX·4, in which X_0, X_1, and X_2 are polynomials in x, is expressible as the sum of two distinct powers of x,

$$f(m)x^h + \phi(m)x^{h+l},$$

where l is a positive integer.

---

* For purposes of simplicity we are using $y = x^m$. If a is a zero of the coefficient of the highest derivative appearing in the linear differential equation, the process of finding the solution in the region of a will usually be simplified by moving the origin to the point a. This is effected by the transformation $x - a = u$.

By the manner in which $f(m)$ and $\phi(m)$ arise, it is obvious that one of them at least must be of degree 2 in m.

I. To find the complementary function we proceed as follows:

(*a*) Suppose that $f(m)$ is of degree 2 in m,[*] so that $f(m) = 0$ has two roots, m_1 and m_2.

Letting $y = c_0x^m + c_1x^{m+l} + \cdots + c_rx^{m+rl} + \cdots$, we have, on substituting in the left-hand member of the differential equation,

$$
(3) \qquad \left\{
\begin{array}{l}
\quad c_0f(m)x^h + c_0\phi(m)x^{h+l} \\
+ c_1f(m+l)x^{h+l} + c_1\phi(m+l)x^{h+2l} \\
+ c_2f(m+2l)x^{h+2l} + c_2\phi(m+2l)x^{h+3l} \\
+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
+ c_{r-1}f(m+[r-1]l)x^{h+(r-1)l} + c_{r-1}\phi(m+[r-1]l)x^{h+rl} \\
+ c_rf(m+rl)x^{h+rl} + c_r\phi(m+rl)x^{h+(r+1)l} \\
+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{array}
\right.
$$

This will be zero if

1° $f(m) = 0$,[†] i.e., if $m = m_1$ or m_2;

2° $c_rf(m+rl) + c_{r-1}\phi(m+[r-1]l) = 0$, for $\left\{\begin{array}{l} r = 1, 2, 3, \cdots \\ m = m_1, m_2 \end{array}\right\}$,

$$
\text{or } c_r = -\frac{\phi(m+[r-1]l)}{f(m+rl)}\,c_{r-1}
$$

$$
= (-1)^r\,\frac{\phi(m+[r-1]l)\phi(m+[r-2]l)\cdots\phi(m+l)\phi(m)}{f(m+rl)f(m+[r-1]l)\cdots f(m+2l)f(m+l)}\,c_0.
$$

To each value of m corresponds, then, in general, a particular integral. If any c_i vanishes, all that follow do so, and that integral appears in finite form.

If the two values of m are equal, of course, the same particular integral will correspond to them. Moreover, if the two m's differ by an integral multiple of l, viz. $m_2 = m_1 + gl$, then, corresponding to the smaller value m_1, the coefficient c_g will be infinite [since $f(m_2) = f(m_1 + gl) = 0$], unless the numerator is also zero. Hence our method gives us, in general, only one particular integral. The remaining integral must then be sought by a modification of our process. See the next two sections.

---

[*] This is always the case when the conditions of part (*a*) of the existence theorem of the previous section hold.

[†] The equation $f(m) = 0$ is known as the *indicial equation.* There is an indicial equation corresponding to each finite point a. The equation $\phi(m) = 0$, used below, is the indicial equation for the point at infinity.

If it should happen that, in the calculation of c_r in terms of c_{r-1}, both numerator and denominator contain a vanishing factor (to the same degree) when the value of a root of the indicial equation is assigned to m, it can be shown that the result obtained by ignoring these factors gives an available value for c_r. (Compare §IX·8.)

If $f(m)$ is of lower degree than 2, the above method may still lead to a series which satisfies the equation. But, the general theory gives us no assurance that it is convergent over a finite region, since the conditions of the existence theorem do not hold in this case.

(*b*) If $\phi(m)$ is of degree 2 in m, $\phi(m) = 0$ will be satisfied by two values of m. Let them be m_1' and m_2'. Putting

$$y = c_0x^m + c_{-1}x^{m-l} + c_{-2}x^{m-2l} + \cdot\cdot\cdot + c_{-r}x^{m-rl} + \cdot\cdot\cdot,$$

we have, on substituting in the left-hand member of the equation,

$$\begin{aligned}
&c_0\phi(m)x^{h+l} + c_0f(m)x^h \\
&+ c_{-1}\phi(m-l)x^h + c_{-1}f(m-l)x^{h-l} \\
&+ c_{-2}\phi(m-2l)x^{h-l} + c_{-2}f(m-2l)x^{h-2l} \\
&+ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
&+ c_{-r+1}\phi(m-[r-1]l)x^{h-(r-2)l} + c_{-r+1}f(m-[r-1]l)x^{h-(r-1)l} \\
&+ c_{-r}\phi(m-rl)x^{h-(r-1)l} + c_{-r}f(m-rl)x^{h-rl} \\
&+ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

This will be zero if

1° $\phi(m) = 0$, i.e., if $m = m_1'$ or m_2';

$$2°\ c_{-r} = -\frac{f(m-[r-1]l)}{\phi(m-rl)}\,c_{-r+1} \text{ for } \begin{Bmatrix} r = 1, 2, 3, \cdot\cdot\cdot \\ m = m_1', m_2' \end{Bmatrix}$$

$$= (-1)^r\frac{f(m-[r-1]l)f(m-[r-2]l)\cdot\cdot\cdot f(m-l)f(m)}{\phi(m-rl)\phi(m-[r-1]l)\cdot\cdot\cdot\phi(m-2l)\phi(m-l)}\,c_0.$$

To each value of m corresponds, in general, a particular integral of the form $x^m(c_0 + c_{-1}x^{-l} + c_{-2}x^{-2l} + \cdot\cdot\cdot + c_{-r}x^{-rl} + \cdot\cdot\cdot)$.

Here x^m is multiplied by a power series in negative ascending powers of x. The convergence of this series, for sufficiently large values of x, is assured by part (*b*) of the existence theorem of the previous section.

If the roots of $\phi(m) = 0$ are equal or differ by a multiple of l, the number of integrals obtained by this method will be less than two, in general. The remaining integral must then be sought by a modification of our process. See the next two sections.

As suggested in the first footnote on p. 187, the method (*a*) enables us to find, in general, two particular integrals for the differential equation, convergent over a nameable region about each of the points a for which the conditions of part (*a*) of the existence theorem of §IX·4 hold.

The method (*b*) gives us, in general, two particular integrals convergent for sufficiently large values of x.

II. To find a particular integral in case the right-hand member is a power of x, viz. Ax^s, we proceed as follows:

If $f(m)$ is of degree 2, we make the following modification of method (*a*):

Equating the first term of (3) identically to Ax^s, we have

$$c_0 f(m)x^h \equiv Ax^s.$$

From this we find

$$h = s \quad \text{and} \quad c_0 f(m) = A. \tag{4}$$

Since h is a linear function of m, the first relation in (4) determines a single value of m, viz. m_s.

Using this value of m in the second relation in (4), we find the required value of c_0.

The remaining coefficients are determined as in (*a*) except that now m_s is used for m.

This method will fail when $f(m_s) = 0$. Then there is no particular integral of the form sought here. This may also be the case for values of s which cause any of the c's to become infinite. Several examples of these occur in some of the exercises in this and the succeeding section.

A corresponding modification of method (*b*) may be used to find a particular integral in descending powers of x, when $\phi(m)$ is of degree 2.

It is beyond the scope of this treatise to give a general method for finding a particular integral.

REMARK. Attention should be called to the fact that the general expressions for c_r in terms of $\phi(m)$ and $f(m)$ given above are not to be memorized, nor even used in solving a given differential equation. They may be used as checks; but, even then, a better check is the substitution of a few of the early terms of the series in the differential equation. It is the principles involved that are to be memorized.

***Exercise* 1.** $(x - x^2)y'' - 3y' + 2y = x + 3x^4.$

Substituting x^m for y in the left-hand member, we have

$$m(m-4)x^{m-1} - (m+1)(m-2)x^m.$$

Here the two distinct powers of x differ by 1, and $m - 1$ is the smaller exponent. Moreover, the coefficient of x^{m-1} is quadratic in m. Hence the method (*a*) applies. Putting

$$y = c_0x^m + c_1x^{m+1} + c_2x^{m+2} + \cdots + c_rx^{m+r} + \cdots,$$

we must have

1° $m(m-4) = 0$, i.e., $m = 0$ or 4;

$$
\begin{aligned}
2^\circ \quad (m+1)(m-3)c_1 &= (m+1)(m-2)c_0 & \therefore c_1 &= \frac{m-2}{m-3}c_0,\\
(m+2)(m-2)c_2 &= (m+2)(m-1)c_1 & \therefore c_2 &= \frac{m-1}{m-2}c_1,\\
(m+3)(m-1)c_3 &= (m+3)mc_2 & \therefore c_3 &= \frac{m}{m-1}c_2,\\
\cdots & & \cdots &\\
(m+r)(m+r-4)c_r &\\
= (m+r)(m+r-3)c_{r-1} & & \therefore c_r &= \frac{m+r-3}{m+r-4}c_{r-1}.
\end{aligned}
$$

For $m = 0$, we have

$$c_1 = \tfrac{2}{3}c_0, \quad c_2 = \tfrac{1}{2}c_1 = \tfrac{1}{3}c_0, \quad c_3 = 0, \quad \therefore c_4 = c_5 = \cdots = 0.$$

Hence

$$c_0(1 + \tfrac{2}{3}x + \tfrac{1}{3}x^2), \quad \text{or} \quad A(3 + 2x + x^2),$$

is an integral. Call it Ay_1.

For $m = 4$, we have

$$c_1 = 2c_0, \quad c_2 = \tfrac{3}{2}c_1 = 3c_0, \quad c_3 = 4c_0, \cdots, c_r = (r + 1)c_0, \cdots.$$

Hence

$$c_0(x^4 + 2x^5 + 3x^6 + \cdots + nx^{n+3} + \cdots)$$

is another integral. Call it By_2.

The complementary function may then be written

$$y = Ay_1 + By_2.$$

The integral y_1, being a polynomial, has no finite singular points. But the power series defining y_2 is convergent only when $|x| < 1$. This is consistent with part (a) of the existence theorem of the previous section, since $x = 1$ is a zero of X_0.

Since $\phi(m)$ is a quadratic in m, the method (b) can also be employed. Putting

$$y = c_0x^m + c_{-1}x^{m-1} + c_{-2}x^{m-2} + \cdots + c_{-r}x^{m-r} + \cdots,$$

we must have

1° $(m+1)(m-2)=0$, i.e., $m=2$ or -1;

$$\begin{aligned}
&2^\circ \; m(m-3)c_{-1}=m(m-4)c_0 && \therefore c_{-1} = \frac{m-4}{m-3}c_0,\\
&(m-1)(m-4)c_{-2}=(m-1)(m-5)c_{-1} && \therefore c_{-2} = \frac{m-5}{m-4}c_{-1},\\
&(m-2)(m-5)c_{-3}=(m-2)(m-6)c_{-2} && \therefore c_{-3} = \frac{m-6}{m-5}c_{-2},\\
&\cdots\cdots\cdots && \cdots\cdots\\
&(m-r+1)(m-r-2)c_{-r} \\
&\qquad = (m-r+1)(m-r-3)c_{-r+1} && \therefore c_{-r} = \frac{m-r-3}{m-r-2}c_{-r+1},\\
&\cdots\cdots\cdots && \cdots\cdots
\end{aligned}$$

For $m = 2$, we have

$$c_{-1} = 2c_0, \quad c_{-2} = 3c_0, \quad c_{-3} = 4c_0, \cdots, \quad c_{-r} = (r + 1)c_0, \cdots.$$

Hence

$$c_0(x^2 + 2x + 3 + 4x^{-1} + 5x^{-2} + \cdots + rx^{3-r} + \cdots)$$

is an integral. Call it $A\bar{y}_1$.

For $m = -1$, we have

$$c_{-1} = \tfrac{5}{4}c_0, \quad c_{-2} = \tfrac{6}{4}c_0, \quad c_{-3} = \tfrac{7}{4}c_0, \cdots, \quad c_{-r} = \frac{r+4}{4}c_0, \cdots.$$

Hence

$$\frac{c_0}{4}[4x^{-1} + 5x^{-2} + 6x^{-3} + \cdots + (r+3)x^{-r} + \cdots]$$

is another integral. Call it $B\bar{y}_2$.

Here $\bar{y}_1 - \bar{y}_2 = y_1$. Since y_1 has no finite singular points, and $\bar{y}_1$ and $\bar{y}_2$ are convergent only when $|x| > 1$, we may replace $\bar{y}_1$ by y_1, and take

$$y = Ay_1 + B\bar{y}_2$$

as the complementary function for $|x| > 1$.

To find a particular integral, we consider each term of the right-hand member separately.

For the term x, using method (*a*), we put $c_0m(m-4)x^{m-1} \equiv x$; whence

$$m = 2 \quad \text{and} \quad c_0 = -\tfrac{1}{4}.$$

For $m = 2$, $$c_r = \frac{r-1}{r-2}c_{r-1}.$$

$\therefore c_1 = 0$, and all the succeeding c's are zero.

Hence $-\frac{1}{4}x^2$ is a particular integral corresponding to x.

Corresponding to $3x^4$, we put $c_0m(m-4)x^{m-1} \equiv 3x^4$; whence

$$m = 5 \quad \text{and} \quad c_0 = \tfrac{3}{5}.$$

For $m = 5$, $$c_r = \frac{r+2}{r+1}c_{r-1}.$$

$$\therefore c_1 = \tfrac{3}{2}c_0 = \tfrac{3}{10}\cdot 3, \quad c_2 = \tfrac{4}{3}c_1 = \tfrac{3}{10}\cdot 4, \quad c_3 = \tfrac{5}{4}c_2 = \tfrac{3}{10}\cdot 5, \cdots.$$

Hence

$$\tfrac{3}{10}[2x^5 + 3x^6 + 4x^7 + \cdots + (n+1)x^{n+4} + \cdots]$$

is a particular integral corresponding to $3x^4$. Comparing this with y_2 above, we see that it is equal to $\frac{3}{10}(y_2 - x^4)$. Hence a particular integral is $-\frac{3}{10}x^4$.

The complete solution is then

$$y = Ay_1 + By_2 - \tfrac{1}{4}x^2 - \tfrac{3}{10}x^4, \text{ for } |x| < 1,$$
$$y = Ay_1 + B\bar{y}_2 - \tfrac{1}{4}x^2 - \tfrac{3}{10}x^4, \text{ for } |x| > 1.$$

These supply a solution holding for each real value of x except 1 and -1.

Since $x = 1$ is a singular point, for which the conditions of part (a) of the existence theorem of the previous section hold, a development in powers of $(x - 1)$ can be found for a second particular integral, in addition to y_1, by method I(a).

As suggested (footnote, p. 186), it will be simpler to introduce the change of independent variable $x - 1 = u$. Then $u = 0$ is the singular point, and we find a solution in increasing powers of u. It will be left as an exercise to show that the transformed equation with right-hand member zero is

$$u(u+1)\frac{d^2y}{du^2} + 3\frac{dy}{du} - 2y = 0.$$

The roots of the indicial equation are 0 and -2. Using the value 0, we find an integral

$$c_0\left(1 + \frac{2}{3}u + \frac{1}{6}u^2\right), \quad \text{or} \quad A(6 + 4u + u^2).$$

Call it Ay_{11}. When expressed in terms of x, this is the same as Ay_1, viz. $A(3 + 2x + x^2)$.

For $m = -2$, we have the integral

$$c_0(u^{-2} + 4u^{-1} + 6 + 4u + u^2), \quad \text{or} \quad c_0(u^{-2} + 4u^{-1} + y_{11}).$$

Hence, the second particular integral may be taken in the form

$$y_{12} = u^{-2} + 4u^{-1} = \frac{1}{(x-1)^2} + \frac{4}{x-1} = \frac{4x-3}{(x-1)^2}.$$

Thus, corresponding to $x = 1$, we have the solution

$$y = A(3 + 2x + x^2) + B\frac{4x-3}{(x-1)^2} - \frac{x^2}{4} - \frac{3x^4}{10}.$$

Here no infinite series are involved. Hence this form of solution, involving only algebraic functions, may be taken as the desired form of solution holding throughout the plane.

As a matter of fact, it is easy to show that

$$y_2 = x^4(1 + 2x + 3x^2 + \cdots) = \frac{x^4}{(1-x)^2} \text{ if } |x| < 1.$$

For,

$$1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x} \text{ if } |x| < 1.$$

Differentiating this power series term by term, we have

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \frac{1}{(1-x)^2} \text{ if } |x| < 1.$$

Similarly,

$$\bar{y}_1 = x^4\left(\frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \cdots\right) = \frac{x^4}{(1-x)^2} \text{ if } |x| > 1,$$

as may be seen by noting that

$$\frac{1}{1-x} = \frac{1}{-x\left(1 - \frac{1}{x}\right)} = -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \cdots \text{ if } |x| > 1;$$

differentiating this Laurent series term by term, we have

$$\frac{1}{(1-x)^2} = \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \cdots \text{ if } |x| > 1.$$

Moreover,

$$\frac{x^4}{(1-x)^2} - (3 + 2x + x^2) = \frac{4x-3}{(1-x)^2}.$$

Usually, when the solution of a differential equation can be expressed in terms of elementary functions, the solution can be found by elementary methods. In this case the differential equation is exact. Hence its solution can be found by quadratures only. The student should do this.

***Exercise* 2.** Find a form for the particular integral of the differential equation of Ex. 1, corresponding to a term x^s in the right-hand member, viz.

$$(x - x^2)y'' - 3y' + 2y = x^s.$$

In the following exercises, find the solutions in powers of x.

3. $y'' + x^2y' + xy = x^s$. **4.** $(x^2 - x)y'' + (x-3)y' - 4y = 0$.

5. $4xy'' + 2y' + y = 0$.

6. $2x(1-x)y'' + (1-x)y' + 3y = 0$.

7. $(2x + x^3)y'' - y' - 6xy = 0$.

8. $9x(1-x)y'' - 12y' + 4y = 0$.

9. $x^2y'' + xy' + (x^2 - n^2)y = 0$, if n is not an integer or zero.

10. $(1 - x^3)y'' - x^2y' + xy = 0$.

11. $(1 - x^2)y'' - 2xy' + 6y = 0$.

12. $x^4y'' + xy' + y = 0.$

13. $2x^2y'' - xy' + (1 - x^2)y = x^2.$

14. $4(x - x^2)y'' + (6 - 8x)y' - y = 0.$

IX·6. Roots of the indicial equation repeated. We shall also introduce the method, applicable in this case, by an illustrative exercise.

Exercise **1.** $xy'' + y' + y = 0.$

The result of putting $y = x^m$ in the left-hand member is

$$m^2x^{m-1} + x^m.$$

Here the two distinct powers of x differ by 1, and $m - 1$ is the smaller exponent. Moreover, the coefficient of x^{m-1} is quadratic in m. Hence the method I(a), §IX·5 applies.

Here the indicial equation is $m^2 = 0$, and its roots are 0, 0. If we put

$$y = c_0x^m + c_1x^{m+1} + c_2x^{m+2} + \cdots,$$

the values of the c's are found to be given by the law

$$c_1 = -\frac{c_0}{(m+1)^2},\ c_2 = -\frac{c_1}{(m+2)^2} = \frac{c_0}{(m+1)^2(m+2)^2}, \cdots,$$

$$c_r = -\frac{c_{r-1}}{(m+r)^2} = (-1)^r\frac{c_0}{(m+1)^2(m+2)^2\cdots(m+r)^2}, \cdots.$$

Hence

$$v = c_0x^m\left[1 - \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2(m+2)^2} - \cdots \right.$$
$$\left. + (-1)^r\frac{x^r}{(m+1)^2(m+2)^2\cdots(m+r)^2} + \cdots\right]$$

is a solution if $m = 0$. We, thus, have one particular integral

$$y_1 = 1 - \frac{x}{(1!)^2} + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \cdots + (-1)^n\frac{x^n}{(n!)^2} + \cdots.$$

To find a second particular integral, we proceed as follows:

The result of substituting the value of v for y in the differential equation is

$$c_0m^2x^{m-1}. \tag{1}$$

The result of substituting the value of $\partial v/\partial m$ for y in the differential equation is

$$(2) \qquad \frac{\partial}{\partial m} c_0 m^2 x^{m-1}, \quad \text{or} \quad 2c_0 m x^{m-1} + c_0 m^2 x^{m-1} \log x.$$

For, since x and m may be taken as independent variables in v, the order of differentiation with respect to them is interchangeable. Thus

$$\begin{aligned} x \frac{d^2}{dx^2} \frac{\partial v}{\partial m} + \frac{d}{dx} \frac{\partial v}{\partial m} + \frac{\partial v}{\partial m} &= x \frac{\partial}{\partial m} \frac{d^2 v}{dx^2} + \frac{\partial}{\partial m} \frac{dv}{dx} + \frac{\partial}{\partial m} v \\ &= \frac{\partial}{\partial m}\left[x \frac{d^2 v}{dx^2} + \frac{dv}{dx} + v \right] = \frac{\partial}{\partial m} c_0 m^2 x^{m-1}. \end{aligned}$$

But (2) also vanishes for $m = 0$. Hence a second integral is

$$\left(\frac{\partial v}{\partial m}\right)_{m=0},$$

where $(\partial v/\partial m)_{m=0}$ is the value of $\partial v/\partial m$ for $m = 0$. Here

$$\begin{aligned} \frac{\partial v}{\partial m} &= v \log x + \\ &2c_0 x^m \left[\left(\frac{1}{m+1}\right) \frac{x}{(m+1)^2} - \left(\frac{1}{m+1} + \frac{1}{m+2}\right) \frac{x^2}{(m+1)^2 (m+2)^2} \right. \\ &\left. + \left(\frac{1}{m+1} + \frac{1}{m+2} + \frac{1}{m+3}\right) \frac{x^3}{(m+1)^2 (m+2)^2 (m+3)^2} - \cdots \right]. \end{aligned}$$

Hence, putting $m = 0$ and letting $y_2 = \dfrac{1}{c_0}\left(\dfrac{\partial v}{\partial m}\right)_{m=0}$, we have

$$\begin{aligned} y_2 = y_1 \log x + 2\left[\frac{x}{(1!)^2} - \left(\frac{1}{1} + \frac{1}{2}\right) \frac{x^2}{(2!)^2} + \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right) \frac{x^3}{(3!)^2} - \cdots \right. \\ \left. + (-1)^{n+1} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \frac{x^n}{(n!)^2} + \cdots \right]. \end{aligned}$$

The general solution is then

$$\begin{aligned} y &= A y_1 + B y_2 \\ &= (A + B \log x) \left[1 - \frac{x}{(1!)^2} + \frac{x^2}{(2!)^2} - \cdots + (-1)^n \frac{x^n}{(n!)^2} + \cdots \right] \\ &\quad + 2B \left[\frac{x}{(1!)^2} - \left(\frac{1}{1} + \frac{1}{2}\right) \frac{x^2}{(2!)^2} + \cdots \right. \\ &\qquad \left. + (-1)^{n+1} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) \frac{x^n}{(n!)^2} + \cdots \right]. \end{aligned}$$

This form is consistent with part (*a*) of the existence theorem of §IX·4.

The process, by which we solved Ex. 1, illustrates the general method:

If the indicial equation has a double root, $m = m_1$, find the form of the integral given by method I(a), §IX·5 for a general value of m. Call this v. Two independent integrals will then be obtained by putting $m = m_1$ in v and in $\partial v/\partial m$ respectively.

Extensions to triple and higher multiple roots of the indicial equation can be made. But, they are beyond the scope of this treatise.

***Exercise* 2.** Find the particular integral which must be added to the complementary function if the right-hand member of the differential equation in Ex. 1 is ax^s, viz.,

$$xy'' + y' + y = ax^s.$$

3. $xy'' + y' + xy = 0.$

4. $(x - x^2)y'' + (1 - 5x)y' - 4y = 0.$

5. $xy'' + (1 + x)y' + 2y = 0.$

IX·7. Two roots of the indicial equation differ by an integral multiple of 1. In order to illustrate the method applicable in this case, we shall consider the following exercise:

***Exercise* 1.** $(x - x^2)y'' - 3xy' - y = 0.$

The result of putting $y = x^m$ in the left-hand member is

$$m(m - 1)x^{m-1} - (m + 1)^2x^m.$$

Here the two distinct powers of x differ by 1, and $m - 1$ is the smaller exponent. The method I(*a*), §IX·5 applies.

The indicial equation is $m(m - 1) = 0$, and its roots are 0, 1. If we put

$$y = c_0x^m + c_1x^{m+1} + c_2x^{m+2} + \cdot \cdot \cdot, \tag{1}$$

the values of the c's are found to be given by the law

$$c_1 = \frac{m+1}{m}c_0,\ c_2 = \frac{m+2}{m+1}c_1 = \frac{m+2}{m}c_0, \cdot \cdot \cdot, c_r = \frac{m+r}{m}c_0, \cdot \cdot \cdot.$$

For $m = 1$, we have one particular integral

$$y_1 = x + 2x^2 + 3x^3 + \cdots.$$

For $m = 0$, all the c's after c_0 are infinite.

To overcome this difficulty, let the first coefficient in the trial series (1) be mk_0. Then

$$c_1 = (m + 1)k_0,\ c_2 = (m + 2)k_0, \cdots, c_r = (m + r)k_0, \cdots,$$

and the result of substituting

$$v = k_0x^m[m + (m + 1)x + (m + 2)x^2 + \cdots + (m + r)x^r + \cdots]$$

for y in the differential equation is

(2) $$k_0m^2(m - 1)x^{m-1}.$$

As in the previous case, the presence of the squared factor m^2 suggests that not only does the expression (2) vanish for $m = 0$, but so also does its derivative with respect to m, viz.

(3) $$2k_0m(m - 1)x^{m-1} + k_0m^2x^{m-1} + k_0m^2(m - 1)x^{m-1}\log x.$$

But, as in the previous case, (3) is the result of substituting $\partial v/\partial m$ for y in the differential equation. Hence $\left(\frac{\partial v}{\partial m}\right)_{m=0}$ is also an integral. Here

$$\frac{\partial v}{\partial m} = v\log x + k_0x^m(1 + x + x^2 + \cdots + x^r + \cdots).$$

Hence, letting $y_2 = \frac{1}{k_0}\left(\frac{\partial v}{\partial m}\right)_{m=0}$, and noting that v reduces to y_1 when $m = 0$,* we have

$$y_2 = y_1\log x + 1 + x + x^2 + \cdots + x^r + \cdots.$$

This illustrates the general method for this case, which may be stated as follows:

If two roots of the indicial equation are m_1 *and* $m_1 + gl$, *where* g *is a positive integer, and* l *is the difference between the exponents of the two distinct powers of* x *arising on substituting* $y = x^m$ *in the differential equation, replace the trial series by*

$$y = (m - m_1)k_0x^m + c_1x^{m+l} + c_2x^{m+2l} + \cdots + c_rx^{m+rl} + \cdots.$$

---

[*] In practice, y_1 should be obtained in this second way. See general method.

Calculate the c's in terms of the coefficient of x^m *by the rule of method* $I(a)$, *§IX·5. Let* v *be the resulting form of* y. *Two independent integrals will then be obtained by putting* $m = m_1$ *in* v *and in* $\partial v/\partial m$ *respectively.*

An integral can also be obtained by putting $m = m_1 + gl$ in v. This differs by a constant factor from the one obtained when $m = m_1$ in v. The latter is, however, more useful in the expression of the other independent particular integral.

Remark. The solution of the differential equation in Ex. 1 can be expressed in terms of elementary functions. As in the case of Ex. 1, §IX·5,

$$x + 2x^2 + 3x^3 + \cdots = x(1 + 2x + 3x^2 + \cdots) = \frac{x}{(1-x)^2},$$

and

$$1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

Hence

$$y_1 = \frac{x}{(1-x)^2} \quad \text{and} \quad y_2 = \frac{x \log x}{(1-x)^2} + \frac{1}{1-x}.$$

Examination of the equation shows it to be exact. Hence, being linear of the second order, its solution can be effected by quadratures. It is suggested that the student do this.

***Exercise* 2.** $x^2y'' + xy' + (x^2 - 1)y = 0.$

3. $xy'' + y = 0.$ **4.** $x^2y'' + xy' + (x^2 - 4)y = 0.$

5. $x^2y'' + x(1 + x)y' + (2x - 1)y = 0.$

6. $(x^2 - x)y'' + (x + 2)y' - 4y = 0.$

IX·8. Gauss's equation. Hypergeometric series. The integration of the equation (reduced to a special form)

$$(Az^2 + Bz + C)\frac{d^2y}{dz^2} + (Dz + E)\frac{dy}{dz} + Fy = 0,$$

where A, B, C, D, E, F are constants and $B^2 - 4AC \neq 0$, leads to a remarkable series exhaustively studied by Karl Friedrich

Gauss (1777–1855). This series and its differential equation were discovered by Euler (1707–1783).

Putting $z = ax + b$, we have

$$[Aa^2x^2 + (2Aab + Ba)x + Ab^2 + Bb + C]\frac{d^2y}{dx^2} + (D'x + E')\frac{dy}{dx} + F'y = 0,$$

where D', E', F' are constants.

Choosing a and b so that*

$$Ab^2 + Bb + C = 0 \quad \text{and} \quad 2Ab + B = -Aa \neq 0,$$

and dividing by Aa^2, we have

$$(x^2 - x)\frac{d^2y}{dx^2} + (Px + Q)\frac{dy}{dx} + Ry = 0,$$

where P, Q, R are constants. If we put

$$P = \alpha + \beta + 1, Q = -\gamma, R = \alpha\beta,$$

our equation takes the form

$$(x^2 - x)\frac{d^2y}{dx^2} + [(\alpha + \beta + 1)x - \gamma]\frac{dy}{dx} + \alpha\beta y = 0.\dagger$$

The result of substituting $y = x^m$ in the left-hand member is

$$-m(m - 1 + \gamma)x^{m-1} + (m + \alpha)(m + \beta)x^m.$$

Using the method I(a) of §IX·5, we have

$$m = 0 \quad \text{or} \quad 1 - \gamma,$$

$$c_1 = \frac{(m + \alpha)(m + \beta)}{(m + 1)(m + \gamma)}c_0,$$

$$c_2 = \frac{(m + \alpha + 1)(m + \beta + 1)}{(m + 2)(m + \gamma + 1)}c_1,$$

$$\cdots\cdots\cdots\cdots$$

$$c_r = \frac{(m + \alpha + r - 1)(m + \beta + r - 1)}{(m + r)(m + \gamma + r - 1)}c_{r-1},$$

$$\cdots\cdots\cdots\cdots$$

---

* If $B^2 - 4AC < 0$ the values of a and b are not real. But this does not affect the form of the equation; and the theory, employing power series, naturally extends itself to the consideration of complex values of coefficients and variables.

† This is usually referred to as *Gauss's equation.*

For $m = 0$, we have

$$c_1 = \frac{\alpha \cdot \beta}{1 \cdot \gamma} c_0, \quad c_2 = \frac{(\alpha+1)(\beta+1)}{2(\gamma+1)} c_1 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} c_0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$c_r = \frac{\alpha(\alpha+1)\cdots(\alpha+r-1)\beta(\beta+1)\cdots(\beta+r-1)}{1 \cdot 2 \cdot 3 \cdots r \cdot \gamma(\gamma+1)\cdots(\gamma+r-1)} c_0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Putting $c_0 = 1$, we have the particular integral

$$y_1 = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \cdots$$
$$+ \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{1 \cdot 2 \cdot 3 \cdots n \cdot \gamma(\gamma+1)\cdots(\gamma+n-1)} x^n + \cdots.$$

This is the *hypergeometric series*, and is usually represented by

$$F(\alpha, \beta, \gamma, x).$$

For $m = 1 - \gamma$, the student should show that the integral is $c_0 y_2$, where

$$y_2 = x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x).$$

If $1 - \gamma$ is not an integer or zero, the general solution is

$$y = Ay_1 + By_2.$$

Moreover, in this case, y_1 reduces to a polynomial if α or β is a negative integer.

If $1 - \gamma = -g$, where g is a positive integer or zero, y_1 is still an integral, but y_2 will be of the form (§§IX·7, IX·6)

$$y_1 \log x + \text{a power series in } x,$$

unless α or β is equal to one of the positive numbers $1, 2, \cdots, g$. In this case, y_2 is also still an integral and reduces to a polynomial if the integer, to which α or β is equal, exceeds 1. Otherwise, if α or $\beta = 1$, a factor in the numerator and one in the denominator of every coefficient, beginning with c_g, vanish. As noted in §IX·5, the result obtained by ignoring these factors (i.e., by taking their quotient equal to 1) gives us an available form for y_2.*

If $1 - \gamma = h$, where h is a positive integer, the expression for y_2 is still an integral, but that for y_1 will contain a logarithmic term (§IX·7), unless α or β is equal to one of the numbers $0, -1, -2, \cdots, -(h-1)$. In this case, y_1 is also still an integral and reduces to a polynomial if the integer, to which α or β is equal, is algebraically greater than $-(h-1)$. Otherwise, a factor in the numerator and one in the denominator of every coefficient, beginning with c_h, vanish. As before, an available form for y_1 is obtained by ignoring these simultaneously vanishing factors in numerator and denominator.

The power series involved in the expressions for y_1 and y_2 are convergent for $|x| < 1$.

Using the method I(*b*) of §IX·5, the student should show that

$$\bar{y}_1 = \frac{1}{x^\alpha} F\left(\alpha, 1 + \alpha - \gamma, 1 + \alpha - \beta, \frac{1}{x}\right),$$

$$\bar{y}_2 = \frac{1}{x^\beta} F\left(\beta, 1 + \beta - \gamma, 1 + \beta - \alpha, \frac{1}{x}\right)$$

are a pair of linearly independent integrals.

A discussion, similar to the one above, can be given here in case $\alpha - \beta$ equals zero, or a positive or negative integer.

The power series involved in the expressions for $\bar{y}_1$ and $\bar{y}_2$ are convergent for $|x| > 1$.

The hypergeometric series may at times represent well-known functions.

Exercise **1.** $F(-n, \beta, \beta, -x) = (1 + x)^n$ for β any constant.

2. $xF(1, 1, 2, -x) = \log(1 + x)$.

3. $\text{Limit}_{\beta\to\infty} F\left(1, \beta, 1, \frac{x}{\beta}\right) = e^x$.

4. $\text{Limit}_{\alpha,\,\beta\to\infty} xF\left(\alpha, \beta, \frac{3}{2}, -\frac{x^2}{4\alpha\beta}\right) = \sin x$.

---

* For example, see Schlesinger, *Einführung in die Theorie der Differentialgleichungen*, §34.

5. $xF(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2) = \sin^{-1} x.$

6. $xF(\frac{1}{2}, 1, \frac{3}{2}, -x^2) = \tan^{-1} x.$

7. $\text{Limit}_{\alpha,\, \beta \to \infty} F\left(\alpha, \beta, \frac{1}{2}, \frac{x^2}{4\alpha\beta}\right) = \cosh x.$

8. Express as hypergeometric series the following functions:
$\frac{1}{1-x}$, $(1+x)^n + (1-x)^n$, $(1+x)^n - (1-x)^n$, $\cos x$, $e^x + e^{-x}$.

For further examples, see Gauss, *Collected Works*, Vol. III, p. 127.

IX·9. Legendre's equation. The differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0,$$

where n is a constant, has been named for Adrien Marie Legendre (1752–1833), an outstanding French mathematician. It occurs frequently in problems of applied mathematics. Often n is a positive integer.

By means of the simple linear transformation of the independent variable mentioned at the beginning of §IX·8, this equation can be brought into the form of a Gauss equation. However, it is easy to find the solution of the equation as power series, either in x or in $1/x$, by the methods of §IX·5. It will be left for the student to show that, by the method I(a), we have

$$y_1 = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + \cdots,$$

$$y_2 = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots;$$

and, by the method I(b), we find

$$\bar{y}_1 = x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2 \cdot 2!(2n-1)(2n-3)}x^{n-4} - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2^3 \cdot 3!(2n-1)(2n-3)(2n-5)}x^{n-6} + \cdots,$$

$$\bar{y}_2 = x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2^2 \cdot 2!(2n+3)(2n+5)}x^{-n-5}$$
$$+ \frac{(n+1)(n+2)(n+3)\cdots(n+6)}{2^3 \cdot 3!(2n+3)(2n+5)(2n+7)}x^{-n-7} + \cdots.$$

The power series defining y_1 and y_2 are convergent if $|x| < 1$. The one defining y_1 reduces to a polynomial if n is an even positive integer, zero, or an odd negative integer. The same is true of the series defining y_2 if n is an odd positive integer or an even negative one.

The infinite series defining $\bar{y}_1$ and $\bar{y}_2$ are convergent if $|x| > 1$.

If n is a positive integer or zero, the series defining $\bar{y}_1$ is finite, coinciding, to within a constant factor, with that defining y_1 if n is even or zero, and with that defining y_2 if n is odd.

Similarly, if n is a negative integer, the series defining $\bar{y}_2$ is finite, coinciding, to within a constant factor, with that defining y_1 or y_2 according as n is odd or even.

Since the roots of the indicial equation are n and $-n-1$, and l is equal to 2, it follows that, if $2n+1$ is zero or an even integer, a new form, containing a logarithmic term, must be found for one of the particular integrals $\bar{y}_1$, $\bar{y}_2$ (§§IX·6, IX·7).

The series representing $\bar{y}_1$ and $\bar{y}_2$ play important rôles in physical applications, when n is a positive integer. Multiplied by definite constants, they are usually designated as follows:

$$P_n = \frac{(2n)!}{2^n(n!)^2}\bar{y}_1 \quad \text{and} \quad Q_n = \frac{2^n(n!)^2}{(2n+1)!}\bar{y}_2.$$

The function P_n is a polynomial of degree n. It is not difficult to show that

$P_0 = 1$, $[n! = 1 \text{ when } n = 0]$ $\qquad P_1 = x$,

$P_2 = \frac{3}{2}x^2 - \frac{1}{2}$, $\qquad P_3 = \frac{5}{2}x^3 - \frac{3}{2}x$,

$P_4 = \frac{7 \cdot 5}{4 \cdot 2}x^4 - 2\frac{5 \cdot 3}{4 \cdot 2}x^2 + \frac{3 \cdot 1}{4 \cdot 2}$, $\qquad P_5 = \frac{9 \cdot 7}{4 \cdot 2}x^5 - 2\frac{7 \cdot 5}{4 \cdot 2}x^3 + \frac{5 \cdot 3}{4 \cdot 2}x$,

. . . .

The function P_n is known as the *Legendrean coefficient of the nth order*, and also as a *zonal harmonic*.

The function Q_n is an infinite series. It is known as a *zonal harmonic of the second kind.*

IX·10. Bessel's equation. Another very important differential equation of Mathematical Physics is

$$x^2y'' + xy' + (x^2 - n^2)y = 0,$$

where n is a constant. The equation is named for the famous German astronomer Friedrich Wilhelm Bessel (1784–1846). In treating a problem in orbital motion, he first made use of functions, now known as *Bessel's functions,* which appear in the solutions of the differential equation under consideration.

Using the method I(a) of §IX·5, it is not difficult to show that, in general, two independent integrals of Bessel's equation are given by

$$y_1 = x^n\left[1 - \frac{x^2}{2^2(1+n)} + \frac{x^4}{2^4 \cdot 2!(1+n)(2+n)} - \frac{x^6}{2^6 \cdot 3!(1+n)(2+n)(3+n)} + \cdots\right],$$

$$y_2 = x^{-n}\left[1 - \frac{x^2}{2^2(1-n)} + \frac{x^4}{2^4 \cdot 2!(1-n)(2-n)} - \frac{x^6}{2^6 \cdot 3!(1-n)(2-n)(3-n)} + \cdots\right].$$

The expression for y_2 may be obtained from that for y_1 by replacing n by $-n$.

If n is neither an integer nor zero,* both series converge for all finite values of x.

Moreover, the series defining y_1 converges when n is a positive integer, and that defining y_2 converges when n is a negative integer.

Bessel's functions are the following constant multiples of these integrals:

$$J_n(x) = \frac{y_1}{2^n\Gamma(1+n)} \quad \text{and} \quad J_{-n}(x) = \frac{y_2}{2^{-n}\Gamma(1-n)},$$

---

* If n is a positive or negative integer or zero, only one of these solutions holds. A second solution may be found by the method of §IX·7 or §IX·6.

where $\Gamma(1+n)$ is the gamma function of Euler. Evidently, J_{-n} is the same function of $-n$ that J_n is of n.

If n is a positive integer, $\Gamma(1+n) = n!$. In this case, we have

$$J_n = \frac{y_1}{2^n n!}.$$

This is known as *Bessel's function of order n.* Taking 0! as equal to 1, we have

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots.$$

IX·11. Riccati's equation. The equation studied by Count Riccati (1676–1754), and to which his name has been given, is of the form

$$\frac{dy}{dx} + by^2 = cx^m,$$

where b, c, m are constants.* The equation in Ex. 1, §IX·3 is of this type. For certain special values of b, c, m this equation can be integrated by means of quadratures.† But, in general, the only way to obtain the solution is to integrate in series.

Riccati equations arise frequently, and it is often desirable to make use of the properties of their solutions without actually knowing the latter. On the other hand, it is sometimes possible to find the general solution by quadratures or by merely algebraic processes, when certain information is available. The following properties will at times prove of value. We shall establish these for a more general form, which is now usually considered as *Riccati's equation*, viz.,

$$\frac{dy}{dx} = X_0 + X_1 y + X_2 y^2,$$

where X_0, X_1, X_2 are functions of x.

---

* Frequently the equation $x\frac{dy}{dx} - \alpha y + \beta y^2 = \gamma x^n$ is taken as the type of a Riccati equation. This is obviously reducible to the other form by the transformation $z = x^a$, $y = uz$.

† See, for example, A. R. Forsyth, *A Treatise on Differential Equations*, 4th Ed., §§108 and following.

1° If a particular integral y_1 is known, the substitution $y = 1/z + y_1$ transforms the equation to

$$\frac{dz}{dx} + (X_1 + 2y_1X_2)z = -X_2,$$

which is linear, and can, therefore, be solved by two quadratures (§II·9). Hence we have the important result that, *if a particular integral y_1 is known for a Riccati equation, the transformation $y = 1/z + y_1$ gives rise to a linear equation in z which can be solved by two quadratures.*

2° Since the form of the solution of a linear equation of the first order is $z = \gamma(x) + C\delta(x)$, that is, since the constant of integration enters linearly, we see that

$$\begin{aligned} y = \frac{1}{z} + y_1(x) &= \frac{1}{\gamma(x) + C\delta(x)} + y_1(x) \\ &= \frac{\alpha(x) + C\beta(x)}{\gamma(x) + C\delta(x)}. \end{aligned}$$

Hence, *the constant of integration appears bilinearly in the general solution of the Riccati equation.*

3° The equation $y = \dfrac{\alpha + \beta C}{\gamma + \delta C}$ determines a particular solution when a definite value is assigned to C. Corresponding to any four values of C, viz. C_1, C_2, C_3, C_4, we have y_1, y_2, y_3, y_4. Since double ratios remain unaltered by bilinear transformations, we see, from the form of the equation defining y, that

$$\{y_1, y_2, y_3, y_4\} = \{C_1, C_2, C_3, C_4\} = \text{a constant}.$$

Hence, *if y_1, y_2, y_3, y_4 are any four particular integrals, the function $\dfrac{(y_4 - y_3)(y_2 - y_1)}{(y_4 - y_1)(y_2 - y_3)}$ is equal to a constant for all values of x.*

4° As a direct consequence of 3°, it follows that, *if we know three particular integrals, y_1, y_2, y_3, the general solution is given at once by*

$$\frac{y - y_3}{y - y_1} = c\,\frac{y_2 - y_3}{y_2 - y_1};$$

whence $y = \dfrac{y_3(y_2 - y_1) - cy_1(y_2 - y_3)}{y_2 - y_1 - c(y_2 - y_3)}$, i.e., *$y$ is given by purely algebraic means.*

5° If y_1 and y_2 are two known particular integrals, let $z = \dfrac{y - y_1}{y - y_2}$. Taking the derivative of the logarithm of both sides, we find

$$\frac{1}{z}\frac{dz}{dx} = \frac{1}{y - y_1}\left(\frac{dy}{dx} - \frac{dy_1}{dx}\right) - \frac{1}{y - y_2}\left(\frac{dy}{dx} - \frac{dy_2}{dx}\right).$$

Since

$$\frac{dy}{dx} = X_0 + X_1 y + X_2 y^2,$$

$$\frac{dy_1}{dx} = X_0 + X_1 y_1 + X_2 y_1^2,$$

$$\frac{dy_2}{dx} = X_0 + X_1 y_2 + X_2 y_2^2,$$

we have

$$\frac{1}{z}\frac{dz}{dx} = X_2(y_1 - y_2);$$

whence

$$z = \frac{y - y_1}{y - y_2} = Ce^{\int X_2(y_1 - y_2)dx},$$

from which y can be obtained at once. Hence, *if two particular integrals y_1 and y_2 are known, the transformation $z = \dfrac{y - y_1}{y - y_2}$ leads to an equation in which the variables are separated, so that it can be solved by a single quadrature.*

Properties 1°, 4°, 5° call attention to the close analogy of the Riccati equation to linear differential equations; for, the knowledge of each additional particular integral brings us nearer to the general solution. Thus, *the knowledge of a single particular integral enables us to reduce the problem of solving a Riccati equation to one of solving a linear equation of the first order, that is, to one involving two quadratures; the knowledge of two particular* *integrals enables us to find the general solution by performing a single quadrature; while a knowledge of three particular integrals gives us the general solution by a very simple algebraic process.*

6° As a matter of fact, *the substitution*

$$y = -\frac{1}{X_2 z}\frac{dz}{dx}$$

transforms the Riccati equation into the homogeneous linear equation of the second order

$$X_2 \frac{d^2z}{dx^2} - (X_1X_2 + X_2')\frac{dz}{dx} + X_0X_2^2z = 0.$$

Let the student show that, conversely, the substitution $y = e^{\int v dx}$ transforms a homogeneous linear equation of the second order into a Riccati equation.

CHAPTER X

TOTAL DIFFERENTIAL EQUATIONS

X·1. Integrable total differential equation. A differential equation, involving three or more variables, of the form

$$(1) \qquad P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0$$

is called a *total differential equation.* We shall be interested primarily in equations whose solutions can be put in the form

$$(2) \qquad u(x, y, z) = c.$$

Such differential equations are said to be *integrable.*

The differential equation arising from (2) as a primitive is

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0.$$

Hence this must either be the same as (1), or differ from it by a factor $\mu(x, y, z)$; i.e., *if* (1) *is integrable, there must be an integrating factor for it.* For, then a function $\mu(x, y, z)$ exists, such that

$$\frac{\partial u}{\partial x} = \mu P, \quad \frac{\partial u}{\partial y} = \mu Q, \quad \frac{\partial u}{\partial z} = \mu R.$$

Moreover,

$$\mu \frac{\partial Q}{\partial z} + Q \frac{\partial \mu}{\partial z} = \mu \frac{\partial R}{\partial y} + R \frac{\partial \mu}{\partial y}, \quad \text{since } \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial z \partial y},$$

$$\mu \frac{\partial R}{\partial x} + R \frac{\partial \mu}{\partial x} = \mu \frac{\partial P}{\partial z} + P \frac{\partial \mu}{\partial z}, \quad \text{since } \frac{\partial^2 u}{\partial z \partial x} = \frac{\partial^2 u}{\partial x \partial z},$$

$$\mu \frac{\partial P}{\partial y} + P \frac{\partial \mu}{\partial y} = \mu \frac{\partial Q}{\partial x} + Q \frac{\partial \mu}{\partial x}, \quad \text{since } \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

The function μ must satisfy these three equations. It is, therefore, necessary that the functions P, Q, R satisfy a certain condition or conditions.

If we multiply these equations by P, Q, R respectively and take their sum, all of the derivatives of μ disappear, and we have

$$(3) \qquad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0,^*$$

after dropping the common factor μ, which is not zero.

This is a necessary condition among the coefficients P, Q, R that (1) be integrable. Moreover, we shall prove that this condition is also sufficient.

Consider any one of the variables, e.g. z, as a constant temporarily. Then the equation (1) takes the form

$$(4) \qquad Pdx + Qdy = 0.$$

Still considering z as constant, we integrate (4); but, now, the constant of integration may involve z. Let the solution be

$$(5) \qquad u(x, y, z) = \phi(z).$$

We shall show that, if the condition (3) is satisfied, we can choose $\phi(z)$ so that (5) will be the solution of (1). For, differentiating (5), taking into account all the variables, we have

$$(6) \qquad \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = d\phi.$$

Since (5) is a solution of (4) when z is taken as constant, we have

$$(7) \qquad \frac{\partial u}{\partial x} = \mu(x, y, z)P, \quad \frac{\partial u}{\partial y} = \mu(x, y, z)Q,$$

where μ is, in fact, an integrating factor of (4). (See §II·6.) Comparing (6) with (1) multiplied by μ, we have

$$(8) \qquad \left(\frac{\partial u}{\partial z} - \mu R\right)dz = d\phi.$$

---

* This may be written in the following symbolic determinant form:

$$(3) \qquad \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0,$$

which is very easy to remember.

This equation can be solved for ϕ provided $\partial u/\partial z - \mu R$ reduces to a function of z and ϕ, i.e., a function of z and u, when use is made of (5). If $\partial u/\partial z - \mu R$ and u are considered as functions of x and y, z being treated as a parameter, the only requirement for $\partial u/\partial z - \mu R$ to be a function of u is the vanishing of their Jacobian:*

$$(9) \qquad \begin{vmatrix} \dfrac{\partial u}{\partial x}, & \dfrac{\partial^2 u}{\partial x \partial z} - \mu \dfrac{\partial R}{\partial x} - R \dfrac{\partial \mu}{\partial x} \\ \dfrac{\partial u}{\partial y}, & \dfrac{\partial^2 u}{\partial y \partial z} - \mu \dfrac{\partial R}{\partial y} - R \dfrac{\partial \mu}{\partial y} \end{vmatrix}.$$

Making use of (7), we may put this in the form

$$\begin{vmatrix} \mu P, & \mu \dfrac{\partial P}{\partial z} + P \dfrac{\partial \mu}{\partial z} - \mu \dfrac{\partial R}{\partial x} - R \dfrac{\partial \mu}{\partial x} \\ \mu Q, & \mu \dfrac{\partial Q}{\partial z} + Q \dfrac{\partial \mu}{\partial z} - \mu \dfrac{\partial R}{\partial y} - R \dfrac{\partial \mu}{\partial y} \end{vmatrix},$$

or

$$\mu^2 \left[P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \right] - \mu R \left(P \frac{\partial \mu}{\partial y} - Q \frac{\partial \mu}{\partial x} \right).$$

If (3) is satisfied, this becomes

$$\mu R \left[\mu \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) + Q \frac{\partial \mu}{\partial x} - P \frac{\partial \mu}{\partial y} \right],$$

or

$$\mu R \left[\frac{\partial(\mu Q)}{\partial x} - \frac{\partial(\mu P)}{\partial y} \right].$$

Since μ is an integrating factor of (4), this vanishes. Hence the Jacobian (9) vanishes, and equation (8) can be solved for ϕ; putting this in (5), we have the solution of our equation (1), and our theorem is proved.

The necessity and sufficiency of the condition (3) can be proved much more briefly as follows (without suggesting, however, a general method for solving a total differential equation when the condition is satisfied):

---

* See Note I in the Appendix.

When integrable, the equation

$$(1) \qquad Pdx + Qdy + Rdz = 0$$

is equivalent to the two partial differential equations

$$(10) \qquad \frac{\partial z}{\partial x} = -\frac{P}{R} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{Q}{R}.$$

For, as we saw above, if

$$\mu(Pdx + Qdy + Rdz) \equiv du(x, y, z),$$

we must have

$$\frac{\partial u}{\partial x} = \mu P, \ \frac{\partial u}{\partial y} = \mu Q, \ \frac{\partial u}{\partial z} = \mu R.$$

And from the solution $u(x, y, z) = c$, we have

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial z}} = -\frac{P}{R} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial z}} = -\frac{Q}{R}.$$

In order that equations (10) may hold simultaneously, it is necessary and sufficient that

$$(11) \qquad \left[\frac{\partial\left(\frac{P}{R}\right)}{\partial y}\right]_{y,z} = \left[\frac{\partial\left(\frac{Q}{R}\right)}{\partial x}\right]_{x,z}.$$

In the notation employed here,

$$\left(\frac{\partial u}{\partial x}\right)_{x,z} \text{ means } \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\,\frac{\partial z}{\partial x}, \qquad \left(\frac{\partial u}{\partial y}\right)_{y,z} \text{ means } \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\,\frac{\partial z}{\partial y}.$$

Compare the author's *Calculus*, §108, p. 346.

Performing the operations implied in (11), we have

$$R\left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\,\frac{\partial z}{\partial y}\right) - P\left(\frac{\partial R}{\partial y} + \frac{\partial R}{\partial z}\,\frac{\partial z}{\partial y}\right) = R\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\,\frac{\partial z}{\partial x}\right) - Q\left(\frac{\partial R}{\partial x} + \frac{\partial R}{\partial z}\,\frac{\partial z}{\partial x}\right).$$

Since $\frac{\partial z}{\partial x} = -\frac{P}{R}$ and $\frac{\partial z}{\partial y} = -\frac{Q}{R}$, this reduces at once to the form (3) above.

X·2. Method of solution. The first proof given in §X·1 for the sufficiency of the condition (3) suggests the following method for solving a total differential equation, which satisfies this condition:

Integrate the equation considering one of the variables as a con-*

---

* We usually select that variable as constant which will have the effect of simplifying the resulting equation in the other two variables as much as possible.

stant. Instead of a constant of integration, introduce an undetermined function ϕ *of this variable. Redifferentiate, taking into account all the variables, and compare the result with the original differential equation. A new differential equation arises, involving only* ϕ *and that variable of which it is a function. From the solution of this equation, the form of* ϕ *is obtained, involving an arbitrary constant. Thus the complete solution is found.*

Apply the test for integrability and integrate the following:

***Exercise* 1.** $y^2dx + zdy - ydz = 0.$

$$\begin{vmatrix} y^2 & z & -y \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & z & -y \end{vmatrix} = y^2(-1-1) + z(0-0) - y(0-2y) = -2y^2 + 2y^2 = 0.$$

While, in this simple case, there is very little choice in selecting the variable to be constant, there is, perhaps, some advantage in letting y be so chosen. Then we have

$$y^2dx - ydz = 0; \quad \text{whence} \quad xy - z = \phi(y).$$

Differentiating this, we have

$$ydx + xdy - dz = d\phi,$$

or

$$y^2dx + xydy - ydz = yd\phi.$$

Comparing this with the original differential equation, we have

$$(xy - z)dy = yd\phi,$$

or

$$\phi dy = yd\phi. \quad \therefore \phi = cy.$$

Hence the general solution is

$$xy - z = cy.$$

2. $zydx - zxdy - y^2dz = 0.$

3. $xdx + ydy - \sqrt{a^2 - x^2 - y^2}dz = 0.$

4. $(x^2 - y^2 - z^2)dx + 2xydy + 2xzdz = 0.$

5. Determine $f(z)$ so that

$$(1 + z^2) \cos x \, dx + f(z)dy + (yz - z \sin x)dz = 0$$

is integrable. When this has been found, solve the equation.

X·3. Special methods of solution.

1° Since an integrable equation is either exact or can be made so by the introduction of an integrating factor, whenever such a factor can be found, by inspection or otherwise, it should be used and the resulting exact equation integrated directly.

In this connection it should be noted that *whenever one of the variables in an integrable total differential equation is "separated" from the other two, the equation is exact.* Thus, z is said to be "separated" from the other variables if P and Q do not involve z, and R does not involve x and y, and, in addition, $R \not\equiv 0$.* The equation is then of the form

$$P(x, y)dx + Q(x, y)dy + R(z)dz = 0, \tag{1}$$

and the condition for integrability reduces to

$$R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \tag{2}$$

Since $R \not\equiv 0$, $\partial P/\partial y - \partial Q/\partial x = 0$; hence, $P(x, y)dx + Q(x, y)dy$ is the differential of some function of x and y. Moreover, $R(z)dz$ is the differential of $\int R(z)dz$. Therefore, (1) is exact.

Thus, in Ex. 1, §X·2, x is separated from the other variables if the equation is divided through by y^2. Doing this, we have

$$dx + \frac{zdy - ydz}{y^2} = 0.$$

This is obviously exact and has for solution

$$x - \frac{z}{y} = c.$$

It is suggested as an exercise that the student examine the remaining exercises of §X·2 for cases in which one of the variables is separable.

2° At times, it is possible to replace a function of two or more of the variables by a single new variable, so that the resulting equation is an ordinary differential equation in two variables, or, at least, takes a simpler form.

---

* If $R \equiv 0$, z does not appear at all, and the equation is an ordinary differential equation involving x and y only.

Thus, in Ex. 4, §X·2, if $y^2 + z^2 = u$, the differential equation becomes the ordinary one

$$x^2dx + xdu - udx = 0.$$

An obvious integrating factor is $1/x^2$; whence the solution is found to be

$$x + \frac{u}{x} = c, \quad \text{or} \quad x^2 + y^2 + z^2 = cx.$$

***Exercise* 1.** $xdx + \sqrt{x^2 - 2z}dy - dz = 0.$

2. $(x + y + z + 1)dx - (x + y + z - 1)dy + dz = 0.$

3. $(x^2 + y + z)dx - xdy - xdz = 0.$

4. $(y^2 + z^2)dx - xydy - xzdz = 0.$

5. $yzdx - xzdy - (x^2 + y^2)dz = 0.$

6. $(y^2z^2 - xyz)dx + x^2zdy + x^2ydz = 0.$

7. $2xydx - zdy - y \log y \cdot dz = 0.$

X·4. Homogeneous equation. If P, Q, and R are homogeneous functions of the same degree, one of the variables can often be separated from the others in a manner analogous to that employed for the homogeneous ordinary differential equation of the first order (§II·2). Here we select two of the variables, x and y for example, and introduce the new variables u and v by the transformation $x = uz$, $y = vz$. Then,

$$dx = zdu + udz, \quad dy = zdv + vdz,$$

and the differential equation becomes

$$z(P_1du + Q_1dv) + (uP_1 + vQ_1 + R_1)dz = 0, \tag{1}$$

where

$$P_1 = P(u, v, 1), \quad Q_1 = Q(u, v, 1), \quad R_1 = R(u, v, 1).$$

If $uP_1 + vQ_1 + R_1 \not\equiv 0$, the equation may be put in the form

$$\frac{P_1du + Q_1dv}{uP_1 + vQ_1 + R_1} + \frac{dz}{z} = 0. \tag{2}$$

Now, if the original equation satisfies the condition for integrability, equation (2) does also; hence the latter is exact, since z is separated from the other variables.

If, however, $uP_1 + vQ_1 + R_1 \equiv 0$,* the problem of solving the total differential equation is replaced by that of solving

$$P_1du + Q_1dv = 0, \tag{3}$$

an ordinary differential equation of the first order.

***Exercise* 1.** $(y^2 + z^2)dx - 2(xy + yz)dy + (x^2 + y^2)dz = 0.$

If we put $x = uy$ and $z = vy$, the equation becomes

$$\frac{(v^2 + 1)du + (u^2 + 1)dv}{(u + v)(uv - 1)} + \frac{dy}{y} = 0, \tag{4}$$

which is exact. Employing the method of §II·8, we must first evaluate

$$\int^{(u)} \frac{(v^2 + 1)du}{(u + v)(uv - 1)},$$

where v is a constant.

This fraction can be decomposed into a sum of partial fractions very easily by adding and subtracting uv in the numerator. Then,

$$\int^{(u)} \left(\frac{v}{uv - 1} - \frac{1}{u + v}\right) du = \log \frac{uv - 1}{u + v}.$$

This is readily seen to be the integral of the entire first term of (4). Hence the solution of (4) is

$$\frac{yuv - y}{u + v} = c.$$

Multiplying numerator and denominator by y, we have finally

$$\frac{xz - y^2}{x + z} = c.$$

---

* This arises when $xP + yQ + zR \equiv 0$, for,

$$xP + yQ + zR \equiv z^{r+1}(uP_1 + vQ_1 + R_1),$$

where r is the common degree of P, Q, and R.

Exercise **2.**

$$z(x^2 + 2xy - y^2)dx + z(2xy + y^2 - x^2)dy - (x^2 + y^2)(x + y)dz = 0.$$

Here $xP + yQ + zR \equiv 0$. Hence, if we put $x = uz$, $y = vz$, the equation becomes

$$(u^2 + 2uv - v^2)du + (v^2 + 2uv - u^2)dv = 0,$$

an ordinary differential equation, not exact, but homogeneous. Hence (§II·11) $1/(uM + vN)$, or $1/(u^2 + v^2)(u + v)$, is an integrating factor. Introducing it, we have

$$\frac{u^2 + 2uv - v^2}{(u^2 + v^2)(u + v)}\,du + \frac{v^2 + 2uv - u^2}{(u^2 + v^2)(u + v)}\,dv = 0. \tag{5}$$

Adding and subtracting u^2 in the numerator of the first term in (5), we have

$$\int^{(u)} \frac{u^2 + 2uv - v^2}{(u^2 + v^2)(u + v)}\,du = \int^{(u)} \left(\frac{2u}{u^2 + v^2} - \frac{1}{u + v}\right)du = \log \frac{u^2 + v^2}{u + v}.$$

This is readily seen to be the integral of the entire left-hand member of (5). Hence the solution of the latter is

$$\frac{u^2 + v^2}{u + v} = c.$$

Multiplying numerator and denominator by z^2, we have finally

$$\frac{x^2 + y^2}{xz + yz} = c.$$

3. $(z^2 - yz)dx + (z^2 + xz)dy + (x^2 + xy)dz = 0.$

4. $2(xy + xz)dx + (y^2 + 2yz - x^2)dy - (x^2 + y^2)dz = 0.$

5. $(y + z)dx + (z - x)dy - (x + y)dz = 0$

6. $xydx - (2x^2 + y^2 + 2z^2)dy + yzdz = 0.$

7. $(y^2 + yz + z^2)dx + (x^2 + xz + z^2)dy + (x^2 + xy + y^2)dz = 0.$

8. $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0.$

9. Taking account of the step from (1) to (2), show that whenever the equation $Pdx + Qdy + Rdz = 0$ is homogeneous, and $xP + yQ + zR \not\equiv 0$, $\dfrac{1}{xP + yQ + zR}$ is an integrating factor.

10. Show that $xP + yQ + zR \equiv 0$ when $Pdx + Qdy + Rdz = 0$ arises from a primitive $u(x, y, z) = c$ in which $u(x, y, z)$ is homogeneous, of degree zero; and conversely.

X·5. Total differential equation involving more than three variables. Consider the equation

$$(1)\qquad Pdx + Qdy + Rdz + Sdt = 0.$$

If this is integrable, it will remain so when we let any of the variables be a constant. Letting x, y, z, t be constants successively, the conditions are

$$(2)\qquad Q\left(\frac{\partial S}{\partial z} - \frac{\partial R}{\partial t}\right) + R\left(\frac{\partial Q}{\partial t} - \frac{\partial S}{\partial y}\right) + S\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) = 0,$$

$$(3)\qquad R\left(\frac{\partial P}{\partial t} - \frac{\partial S}{\partial x}\right) + S\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + P\left(\frac{\partial S}{\partial z} - \frac{\partial R}{\partial t}\right) = 0,$$

$$(4)\qquad S\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) + P\left(\frac{\partial S}{\partial y} - \frac{\partial Q}{\partial t}\right) + Q\left(\frac{\partial P}{\partial t} - \frac{\partial S}{\partial x}\right) = 0,$$

$$(5)\qquad P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0.$$

But these conditions are not all independent. For, if we multiply (2), (3), (4) by P, $-Q$, R respectively and add, we obtain (5) multiplied by S. Moreover, it can be shown that, in general, any three of these conditions are independent.

If the equation contains n variables, we obtain as many conditions as there are ways of selecting three variables from n; that is, the number of conditions is $n(n-1)(n-2)/3!$. Of these, only $(n-1)(n-2)/2$ are independent.

Moreover, these conditions can also be shown to be sufficient. When they hold, the solution is found for an equation in n variables by successive applications of the method for one in three variables. First integrate, considering all but two of the variables constant. Replace the constant of integration by a function ϕ of those variables temporarily considered constant. Redifferentiate, taking into account all the variables, and compare the result with the given equation. The two variables, originally treated differently from the others, no longer appear in the new differential equation, which is integrable and involves, at most, the remaining $n-2$ variables and ϕ, that is,

$n - 1$ variables in all. The process may then be repeated. The following exercise will illustrate.

Exercise 1. $z(y+z)dx + z(t-x)dy + y(x-t)dz + y(y+z)dt = 0.$

Let y and z be constants temporarily. Integrating, we have

$$xz + yt = \phi(y, z).$$

Differentiating and comparing with the original equation, we have

$$(ty + zx)(dy + dz) = (y + z)d\phi,$$

or

$$\phi(dy + dz) = (y + z)d\phi.$$

We now have an equation in the three variables y, z, ϕ. This can be solved by the general method (§X·2). But an obvious integrating factor is $1/(y + z)\phi$. Introducing this, we have

$$\phi = c(y + z).$$

Hence the general solution is

$$xz + yt = c(y + z).$$

Special methods, corresponding to those of §§X·3 and X·4, can also be employed, at times.

Thus, in the above exercise, since the equation is homogeneous, let

$$x = ut,\ y = vt,\ z = wt.$$

The resulting equation is

$$\frac{w(v+w)du + w(1-u)dv + v(u-1)dw}{(v+w)(uw+v)} + \frac{dt}{t} = 0, \tag{6}$$

which is exact. If we write the respective coefficients of dv and dw as

$$w(1-u) = w - wu + v - v = v + w - (uw + v)$$

and

$$v(u-1) = uv - v + uw - uw = u(v+w) - (uw+v),$$

equation (6) may be written

$$\frac{wdu + dv + udw}{uw + v} - \frac{dv + dw}{v + w} + \frac{dt}{t} = 0.$$

Its solution is

$$\frac{t(uw + v)}{v + w} = c,$$

or, finally,

$$xz + yt = c(y + z).$$

2. $xtdx + ztdy + ytdz + (t + 1)(x^2 + 2yz)dt = 0.$

3. $(yt - yz - zt)dx + (x + t)(x + z)dy + (x + t)(y + t)dz$
$+ (xz - xy - yz)dt = 0.$

4. $(y + z - t)dx - (x + 2z - t)dy - (x - 2y + t)dz$
$+ (x - y + z)dt = 0.$

X·6. Non-integrable case. If $Pdx + Qdy + Rdz = 0$ does not satisfy the condition for integrability, it is impossible to find its general solution in the form

$$u(x, y, z) = c.$$

But, as the equation involves three variables, we should expect to find an indefinite number of solutions. As a matter of fact, if we assume an arbitrary relation, $\psi(x, y, z) = 0$, this will determine one of the variables in terms of the other two. Using this relation in the original equation, we obtain a new differential equation in two variables. We see, then, that *the general solution of a total differential equation, which does not satisfy the condition for integrability, consists of an arbitrarily chosen relation among the variables and a second relation involving an arbitrary constant.* The latter relation depends upon the choice of the former, and cannot be determined until the choice has been made.

Exercise. $ydx + xdy - (x + y + z)dz = 0.$

This, it is readily seen, does not satisfy the condition for integrability. If we assume $x + y + z = 0$, our equation becomes

$$ydx + xdy = 0, \text{ whose solution is } xy = c.$$

Hence a solution is $\begin{cases} x + y + z = 0, \\ \qquad\quad xy = c. \end{cases}$

If we assume $x + y = 0$, our equation becomes $ydx + xdy - zdz = 0$, whose solution is $2xy - z^2 = c$. Hence another solution is

$$\begin{cases} x + y = 0, \\ 2xy - z^2 = c. \end{cases}$$

X·7. Geometrical interpretation. It will be recalled that an equation in three variables

$$(1) \qquad u(x, y, z) = c$$

represents a surface in space of three dimensions. If (x, y, z) is any point on this surface, the direction cosines of the normal to this surface at that point are proportional to the values which the respective partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial z}$$

take at that point. If (1) is the general solution of the integrable total differential equation

$$(2) \qquad Pdx + Qdy + Rdz = 0,$$

we saw, in §X·1, that $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$ are proportional to P, Q, R respectively. Hence the direction cosines of the normal to the surface (1), at any point on it, are proportional to the values which P, Q, R have at that point. We shall restrict our considerations to a region in which the functions P, Q, R are regular and do not vanish simultaneously. At each point (x, y, z) in this region P, Q, R have definite values. These may be taken as proportional to the direction cosines of a line through that point. The geometrical interpretation of the fact that (1) is the general solution of (2) may now be stated as follows: At each point in the region of space under consideration, the differential equation (2) determines a direction. The solution (1) is the equation of a family of surfaces, known as *integral surfaces,* whose normal at each point is the line determined by the differential equation at that point.

In the non-integrable case, the assumed relation

$$\psi(x, y, z) = 0 \tag{3}$$

is the equation of a surface, the direction cosines of whose normal at each point are proportional respectively to the values of

$$\frac{\partial \psi}{\partial x}, \ \frac{\partial \psi}{\partial y}, \ \frac{\partial \psi}{\partial z}$$

at that point. In general, these are not proportional to P, Q, R at that point. Thus, at each point of the surface (3) we have two lines, the normal to the surface and the line determined by the differential equation (2). The solution in this case consists of two relations, viz. the assumed relation (3) and another one involving an arbitrary constant. Since two equations in three variables determine a curve, and, since one of the two equations, constituting the solution, is the relation (3), we see that the geometrical representation of the solution in this case is a family of curves on the assumed surface (3). Moreover, it can be shown that the line determined by the differential equation (2), at each point of the region under consideration, is cut orthogonally by the curve of the family through that point. Thus, in the case of Exercise, §X·6, we have on the plane $x + y + z = 0$ the curves cut out upon it by the family of cylinders $xy = c$; while on the plane $x + y = 0$ we have the curves cut out by the hyperbolic paraboloids $2xy - z^2 = c$.

This applies also to an integrable equation, except that, in this case, the curves on an arbitrarily selected surface are precisely the curves of intersection of this surface with the family of integral surfaces.

X·8. Summary. The general method for solving the total differential equation

$$Pdx + Qdy + Rdz = 0,$$

satisfying the condition for integrability, requires the solution of two ordinary differential equations of the first order (§X·2).

The problem of solving such an equation can often be simplified.

Since an integrating factor exists, if one can be found by inspection, it should be employed, thus reducing the problem of solving the equation to quadratures (§X·3).

If one of the variables can be separated from the others, an integrating factor can be found (§X·3).

If P, Q, and R are homogeneous and of the same degree, the problem of solution can be reduced either to quadratures or to the solution of a single ordinary differential equation of the first order (§X·4).

At times, it is possible to introduce a new variable which transforms the total differential equation into an ordinary one of the first order (§X·3).

If the condition for integrability is not satisfied, solutions can still be found, but of a more special type (§X·6).

The general method for solving a total differential equation in n variables, satisfying the conditions for integrability, requires the solution of $n - 1$ ordinary differential equations of the first order (§X·5).

The problem of solving such an equation can often be simplified by methods analogous to those applicable to the equation when $n = 3$.

Apply the test for integrability, and solve the following:

***Exercise* 1.** $(y + z)dx + (z + x)dy + (x + y)dz = 0.$

2. $(z + 1)(xdx + ydy) - (x^2 + y^2)dz = 0.$

3. $(x + y^2 + z^2 + 1)dx + 2ydy + 2zdz = 0.$

4. $(y + a)^2dx + zdy - (y + a)dz = 0.$

5. $(y + z)dx + dy + dz = 0.$

6. $2xdx + dy + (2x^2z + 2yz + 2z^2 + 1)dz = 0.$

7. $(2x + y^2 + 2xz)dx + 2xydy + x^2dz - dt = 0.$

8. $zxdy - yzdx + x^2dz = 0.$

9. $x(y - 1)(z - 1)dx + y(z - 1)(x - 1)dy + z(x - 1)(y - 1)dz = 0.$

10. $(y - z)dx + 2(x + 3y - z)dy - 2(x + 2y)dz = 0.$

11. $t(y + z)dx + t(y + z + 1)dy + tdz - (y + z)dt = 0.$

12. $z(y + z)dx + z(t - x)dy + y(x - t)dz + y(y + z)dt = 0.$

13. $(yz - xyz)dx + (zx - xyz)dy + (xy - xyz)dz = 0.$

14. $(ly - kz)dx + (hz - lx)dy + (kx - hy)dz = 0.$

15. $yzdx - xtdy + xy \log x \cdot dz - xy \log y \cdot dt = 0.$

16. $(y^4 + z^4 + t^4)dx - 4(x^2zt + xy^3)dy$
$-4(x^2yt + xz^3)dz - 4(x^2yz + xt^3)dt = 0.$

CHAPTER XI

SYSTEMS OF SIMULTANEOUS EQUATIONS

XI·1. General method of solution. It is proved in the general theory of ordinary differential equations that a system of n equations involving n dependent variables can, in general, be solved.

We shall consider here the case of $n = 2$.* Let the equations be

$$f_1[(x)_m, \quad (y)_r, t] = 0, \tag{1}$$

$$f_2[(x)_{m+p}, (y)_s, t] = 0, \tag{2}$$

where the highest ordered derivatives of x appearing in (1) and (2) are respectively m and $m + p$, those of y are r and s. Here t is the independent variable. Some of the numbers m, p, r, s may be zero. By differentiating these two equations we obtain additional ones, involving higher derivatives of the dependent variables. This process can be conducted in such a manner that a stage is reached at which the number of equations exceeds by two the order of the highest derivative of one of the variables. This dependent variable and all its derivatives can, in general, be eliminated from these equations. The result is a single differential equation involving only one dependent variable. Thus, differentiating (1) p times, we obtain successively

$$f_3[(x)_{m+1}, (y)_{r+1}, t] = 0, \tag{3}$$

$$f_4[(x)_{m+2}, (y)_{r+2}, t] = 0, \tag{4}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f_{p+2}[(x)_{m+p}, (y)_{r+p}, t] = 0. \tag{$p+2$}$$

---

* With obvious modifications, this method can be made to apply in the case of n any positive integer.

We now have $p + 2$ equations from which to eliminate x and all of its $p + m$ derivatives. In general, this will not be sufficient,* for we must have at our disposal one more equation than the number of quantities to be eliminated. We proceed now to differentiate both (2) and ($p + 2$). Since this introduces two new equations and only one new derivative of x, we see that, by repeating the process the proper number of times, the number of equations will exceed by unity that of the quantities to be eliminated. Performing the elimination, we have a single equation in y. Integrating this and substituting the value of y in (1), we have an equation in x only, which must then be solved. If $m > 0$, additional constants of integration appear which, in general, are not independent of those appearing in the general solution of y. The relations between the constants can be determined by making use of the fact that the values found for x and y must satisfy equation (2) identically. This will be illustrated in the next section.

Of course, if preferred, one may first eliminate y and its derivatives, and then solve for x.

Or, one may solve for x and y separately. The relations between the constants of integration can then be found by substituting the general solutions for x and y in one, or, if necessary, in both of the equations (1) and (2).

---

* It will be sufficient if $m = 0$, or if, during the process, a set of differential equations arises from which x or some of its derivatives are absent, so that those which do appear can be eliminated from a set of fewer equations. Thus, consider the system

$$A\frac{d^2x}{dt^2} + By = 0 \quad \text{and} \quad C\frac{dx}{dt} + D\frac{d^2y}{dt^2} = 0,$$

where A, B, C, D are constants. Differentiating the second equation, we obtain

$$C\frac{d^2x}{dt^2} + D\frac{d^3y}{dt^3} = 0,$$

giving us three equations from which $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ can be eliminated. The resulting differential equation in y is of the third order, and not of the fourth order, as the general method would lead us to expect. Here, however, the general solution of the system of equations involves four arbitrary constants.

The differential equations, resulting from the elimination of x and its derivatives, and of y and its derivatives respectively, are frequently of the same order. Then the number of independent constants appearing in the general solution of the system of equations is equal to this common order. If the orders of the two resulting equations are different, the number of independent constants is equal to the higher of the two orders.

XI·2. System of linear equations with constant coefficients. The general method of the previous section can be carried out readily in case the equations are linear and the coefficients are constants. Thus, consider the system

$$\frac{dx}{dt} - \frac{dy}{dt} + x = \cos t,$$

$$\frac{d^2x}{dt^2} - \frac{dy}{dt} + 3x - y = e^{2t}.$$

These may be written

$$(D + 1)x - Dy = \cos t, \tag{1}$$

$$(D^2 + 3)x - (D + 1)y = e^{2t}, \tag{2}$$

where $D \equiv d/dt$. Differentiating (1), we have

$$(D^2 + D)x - D^2y = -\sin t. \tag{3}$$

We must eliminate x, Dx, D^2x; this requires four equations. Hence we must differentiate (2) and (3). This gives rise to

$$(D^3 + 3D)x - (D^2 + D)y = 2e^{2t}, \tag{4}$$

$$(D^3 + D^2)x - D^3y = -\cos t. \tag{5}$$

We now have five equations, from which we can eliminate the four quantities x, Dx, D^2x, D^3x. By taking $-3 \times (1)$, $1 \times (2)$, $1 \times (4)$, $-1 \times (5)$, and adding the results, we have

$$(D^3 - D^2 + D - 1)y = 3e^{2t} - 2\cos t, \tag{6}$$

which is a linear equation in y only, and can be solved readily.

Before doing so, however, we shall see how (6) can be obtained directly from (1) and (2). Since, when looked upon algebraically, (3) and (5) are respectively D and D^2 times (1), and (4) is D times (2), the above method of elimination is equivalent to subtracting $(D^2 + 3)$ times (1) from $(D + 1)$ times (2). But, this is precisely the method we would have pursued in eliminating x from (1) and (2), had D and its powers been algebraic quantities instead of operators. Now, so long as the equations are linear with constant coefficients, this process is always allowable, since it involves only the operations of addition, subtraction, and multiplication with the operator D. Hence we need only write our equations in the form of (1) and (2) and solve them as algebraic equations, remembering, however, that D is an operator in case there are any terms in the right-hand members. In practice, it is frequently convenient to use determinants. Thus, solving (1) and (2) for y, we have

$$\begin{vmatrix} D+1 & -D \\ D^2+3 & -(D+1) \end{vmatrix} y = \begin{vmatrix} D+1 & \cos t \\ D^2+3 & e^{2t} \end{vmatrix},$$

or

$$(D^3 - D^2 + D - 1)y = 3e^{2t} - 2\cos t. \tag{6}$$

The complementary function is $Y = c_1e^t + c_2 \sin t + c_3 \cos t$.

For the particular integral, we try $V = ae^{2t} + bt \sin t + ct \cos t$. Substituting this for y in (6), we find that $a = \frac{3}{5}$, $b = \frac{1}{2}$, $c = \frac{1}{2}$; whence

$$y = c_1e^t + c_2 \sin t + c_3 \cos t + \tfrac{3}{5}e^{2t} + \tfrac{1}{2}t \sin t + \tfrac{1}{2}t \cos t. \tag{7}$$

To find x we may substitute this value in either (1) or (2), and solve the resulting equation in x.*

---

* In general, in solving for the variable first eliminated, it is necessary to solve a differential equation. For example, if we put the value of y given by (7) in (1), we have an equation of the first order to solve; if we put it in (2), we have an equation of the second order to solve. The new constants of integration that arise now are not arbitrary, but must be determined so that the other equation may also be satisfied. This is done by substituting in the other equation, and equating coefficients. In this particular exercise it would have been simpler to solve for x first. The value of y could then be obtained immediately from the equation resulting from subtracting (1) from (2).

Or, we can eliminate y and its derivatives, and obtain a linear equation in x. Thus,

$$\begin{vmatrix} D+1 & -D \\ D^2+3 & -(D+1) \end{vmatrix} x = \begin{vmatrix} D & \cos t \\ D+1 & e^{2t} \end{vmatrix},$$

or

$$(D^3 - D^2 + D - 1)x = 2e^{2t} + \sin t - \cos t.^* \tag{8}$$

The solution of (8) is

$$x = c_1'e^t + c_2' \sin t + c_3' \cos t + \tfrac{2}{5}e^{2t} + \tfrac{1}{2}t \cos t. \tag{9}$$

These constants are not independent of those in (7). Their values may be found by substituting (7) and (9) in either of the original equations and equating coefficients. Doing this, we find $c_1' = \frac{1}{2}c_1$, $c_2' = \frac{1}{2}(c_2 - c_3) + \frac{3}{4}$, $c_3' = \frac{1}{2}(c_2 + c_3) + \frac{1}{4}$.

Hence the general solution of our system of equations is

$$\begin{aligned} 4x &= 2c_1e^t + (2c_2 - 2c_3 + 3) \sin t + (2c_2 + 2c_3 + 1) \cos t \\ &\qquad + \tfrac{8}{5}e^{2t} + 2t \cos t, \\ y &= c_1e^t + c_2 \sin t + c_3 \cos t + \tfrac{3}{5}e^{2t} + \tfrac{1}{2}t (\sin t + \cos t). \end{aligned}$$

***Exercise* 1.** $\begin{cases} 3\dfrac{dx}{dt} + 3x + 2y = e^t, \\ 4x - 3\dfrac{dy}{dt} + 3y = 3t. \end{cases}$

2. $\begin{cases} 2\dfrac{d^2y}{dt^2} - \dfrac{dx}{dt} - 4y = 2t, \\ 4\dfrac{dx}{dt} + 2\dfrac{dy}{dt} - 3x = 0. \end{cases}$

3. $\begin{cases} \dfrac{d^2x}{dt^2} - 3x - 4y = \sin t, \\ \dfrac{d^2y}{dt^2} + x + y = \cos t. \end{cases}$

---

* We see by this method of solution that the differential equations in x and in y, each resulting from the elimination of the other variable, have the same left-hand members, and that the complementary functions are, therefore, of the same form. It is obvious that this is also true in the case of n dependent variables defined by n linear equations with constant coefficients.

4. $$\begin{cases} \dfrac{d^2x}{dt^2} - \dfrac{d^2y}{dt^2} + x - y = \sin \omega t, \\ 2\dfrac{dx}{dt} + \dfrac{dy}{dt} + y = 0. \end{cases}$$

XI·3. System of equations of the first order. If the equations are of the first order, we shall suppose them solved for the first derivatives of each of the dependent variables. [We shall consider the case of two dependent variables. But the methods, developed here, obviously apply to the case of n such variables.] Let the system be

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{P(x, y, t)}{R(x, y, t)}, \\ \dfrac{dy}{dt} = \dfrac{Q(x, y, t)}{R(x, y, t)}. \end{cases} \tag{1}$$

The general method of §XI·1 applies. But, in certain cases, the solution can be obtained very much more readily by special methods. Some of these cases will be considered in this section. First, we shall write equations (1) in the more symmetrical form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dt}{R}. \tag{2}$$

1° One of these equations may involve only two of the variables, or it may be possible, by a proper choice of a pair of members of (2), to eliminate a common factor so as to obtain an equation involving only two of the variables. Thus, for example, suppose that t does not appear in $dx/P = dy/Q$, or can be removed from it. Solving this, we have

$$u(x, y) = c_1. \tag{3}$$

If this can be done a second time, so that a second relation of the form

$$v(y, t) = c_2 \tag{4}$$

can be found, the complete solution consists of the two relations (3) and (4).

REMARK. From the general method of §XI·1, we see that the general solution of (1) or (2) consists of two relations between the variables involving two arbitrary constants,

$$x = f(t, c_1, c_2), \ y = \phi(t, c_1, c_2).$$

The solutions found in this section are of the form

$$u(x, y, t) = c_1, \ v(x, y, t) = c_2.$$

It should be noted that there is no unique form for the functions u and v. We shall see in §XIII·1 that

$$\phi(u, v) = \text{const.},$$

where ϕ, an arbitrarily selected function of u and v, may replace either one of the two relations in the general solution.

***Exercise* 1.** $\dfrac{dx}{yt} = \dfrac{dy}{tx} = \dfrac{dt}{xy}$.

From the first two members we have $u \equiv x^2 - y^2 = c_1$.

From the last two members we have $v \equiv y^2 - t^2 = c_2$.

(Using the first and last members, we obtain $t^2 - x^2 = c$. But this is obviously not distinct from the other two.)

2° If we can find one integral expression by the previous method, e.g. (3), we can, by means of it, express one of the variables in terms of c_1 and the other variable; thus, we can solve (3) for x in terms of c_1 and y. Substituting this value of x in $dy/Q = dt/R$, we have an equation involving y, t, and the constant c_1. Solving this, we have a second relation

$$w(y, t, c_1) = c_2. \tag{5}$$

Relations (3) and (5) together constitute the general solution. At times, it is desirable to replace c_1 in (5) by its value in terms of x and y. The solution is then

$$u(x, y) = c_1, \ w(y, t, u) = c_2.$$

***Exercise* 2.** $\dfrac{dx}{xt} = \dfrac{dy}{yt} = \dfrac{dt}{xy}$.

Here we have

$$\frac{dx}{x} = \frac{dy}{y}; \quad \text{whence} \quad \frac{x}{y} = c_1.$$

Therefore, $x = c_1y$; and we have

$$\frac{dy}{yt} = \frac{dt}{c_1y^2}, \quad \text{or} \quad c_1y\,dy = t\,dt;$$

whence

$$c_1y^2 - t^2 = c_2, \quad \text{or} \quad xy - t^2 = c_2.$$

The solution is

$$x - c_1y = 0,\ xy - t^2 = c_2.$$

3° It happens sometimes that multipliers $\lambda(x, y, t)$, $\mu(x, y, t)$, $\nu(x, y, t)$ can be found to enable us to make use of the fact that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dt}{R} = \frac{\lambda dx + \mu dy + \nu dt}{\lambda P + \mu Q + \nu R}.$$

(*a*) The last member, when combined with one of the others, may give rise to an equation which can be solved.

(*b*) By a choice of two sets of multipliers,

$$\frac{\lambda_1 dx + \mu_1 dy + \nu_1 dt}{\lambda_1 P + \mu_1 Q + \nu_1 R} = \frac{\lambda_2 dx + \mu_2 dy + \nu_2 dt}{\lambda_2 P + \mu_2 Q + \nu_2 R}$$

may be solvable.

(*c*) $\lambda P + \mu Q + \nu R$ may equal zero, at the same time that $\lambda dx + \mu dy + \nu dt = 0$ satisfies the condition for integrability (§X·1).

If we can find two independent relations by any of these methods, each involving an arbitrary constant, we have the general solution.

***Exercise* 3.** $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dt}{t}.$

From $\dfrac{dx}{y} = \dfrac{dy}{x}$, we have $x^2 - y^2 = c_1$.

Letting $\lambda = \mu = 1, \nu = 0$, we have

$$\frac{dx + dy}{x + y} = \frac{dt}{t}; \quad \text{whence} \quad x + y = c_2t.$$

By letting $\lambda = 1, \mu = -1, \nu = 0$, another relation

$$(x - y)t = c_3$$

can be found. But this is not independent of the other two.

***Exercise* 4.** $\dfrac{dx}{x^2+y^2} = \dfrac{dy}{2xy} = \dfrac{dt}{(x+y)t}$.

Letting $\lambda_1 = 1$, $\mu_1 = 1$, $\nu_1 = 0$, and $\lambda_2 = 1$, $\mu_2 = -1$, $\nu_2 = 0$, we have

$$\frac{dx+dy}{(x+y)^2} = \frac{dx-dy}{(x-y)^2}; \quad \text{whence} \quad \frac{1}{x+y} = \frac{1}{x-y} - c_1;$$

or

$$2y = c_1(x^2 - y^2).$$

Again,

$$\frac{dx+dy}{(x+y)^2} = \frac{dt}{(x+y)t}; \quad \text{whence} \quad x + y = c_2t.$$

***Exercise* 5.** $\dfrac{dx}{cy-bt} = \dfrac{dy}{at-cx} = \dfrac{dt}{bx-ay}$.

Letting $\lambda = a$, $\mu = b$, $\nu = c$, we have $\lambda P + \mu Q + \nu R = 0$.

$$\therefore\ adx + bdy + cdt = 0; \quad \text{whence} \quad ax + by + ct = c_1.$$

Similarly, letting $\lambda = x$, $\mu = y$, $\nu = t$, we have

$$xdx + ydy + tdt = 0; \quad \text{whence} \quad x^2 + y^2 + t^2 = c_2.$$

6. $\dfrac{dx}{y+1} = \dfrac{dy}{x+1} = \dfrac{dt}{t}$.

7. $\dfrac{dx}{-y} = \dfrac{dy}{x} = \dfrac{dt}{1+t^2}$.

8. $\dfrac{dx}{yt} = \dfrac{dy}{xt} = \dfrac{dt}{x+y}$.

9. $\dfrac{dx}{x^2-y^2-t^2} = \dfrac{dy}{2xy} = \dfrac{dt}{2xt}$.

10. $\dfrac{dx}{y+t} = \dfrac{dy}{t+x} = \dfrac{dt}{x+y}$.

11. $\dfrac{dx}{x^2+y^2+yt} = \dfrac{dy}{x^2+y^2-xt} = \dfrac{dt}{(x+y)t}$.

12. $\dfrac{dx}{y^3x-2x^4} = \dfrac{dy}{2y^4-x^3y} = \dfrac{dt}{9t(x^3-y^3)}$.

13. $\dfrac{dx}{x(y^2-t^2)} = \dfrac{dy}{y(t^2-x^2)} = \dfrac{dt}{t(x^2-y^2)}$.

14. $\dfrac{dx}{y^2-t^2} = \dfrac{dy}{-3x^2t-xy} = \dfrac{dt}{3x^2y+xt}$.

15. $\dfrac{xdx}{t^2-2yt-y^2} = \dfrac{dy}{y+t} = \dfrac{dt}{y-t}$.

16. $\dfrac{dx}{xt(xy+t^2)} = \dfrac{dy}{-yt(xy+t^2)} = \dfrac{dt}{x^4}$.

XI·4. Geometrical interpretation. The functions $P(x, y, z)$,* $Q(x, y, z)$, $R(x, y, z)$ may be regarded as determining a line through each point (x, y, z), at which they are regular and do not vanish simultaneously (see §X·7). An integral curve of the system $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ is, then, a curve which, at each of its points, is tangent to the line determined by the values of P, Q, R at that point. The general solution, we have seen, consists of two relations, $u(x, y, z) = c_1$, $v(x, y, z) = c_2$, involving two arbitrary constants. That is, the general solution represents a doubly infinite system of curves, which are the intersections of two singly infinite systems of surfaces.

The family of integral curves is fixed by the differential equations, but the families of surfaces determining them are not. If $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are such a pair of families of surfaces, we may employ in their stead the pair $\phi(u, v) = a$ and $\psi(u, v) = b$, where ϕ and ψ are any two arbitrarily selected independent functions of u and v. This follows from the Remark in §XI·3. Thus, in the case of Ex. 3, §XI·3, the integral curves are the intersections of the family of cylinders $x^2 - y^2 = c_1$ with the family of planes $x + y - c_2z = 0$. They may also be obtained as the intersections of the family of cylinders $xz - yz = c_1$ with the family of planes $x + y - c_2z = 0$.

In §X·7 we saw that a solution of the integrable total differential equation $Pdx + Qdy + Rdz = 0$ represents a surface whose normal, at each of its points, has its direction cosines proportional to the values of P, Q, R at that point. Hence we see that the integral curves of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ cut orthogonally any integral surface of $Pdx + Qdy + Rdz = 0$. Since we can find a family of integral surfaces of $Pdx + Qdy + Rdz = 0$ only when P, Q, R satisfy the condition for integrability [§X·1, (3)], we see that only in this case will there exist a family of surfaces which has the family of integral curves of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ as orthogonal trajectories. Thus, since $yzdx + zxdy + xydz = 0$ has $xyz = c$ as its general solution, we see from Ex. 1, §XI·3, that the curves of intersection of the cylinders $x^2 - y^2 = c_1$ and $y^2 - z^2 = c_2$ are cut orthogonally by the family of surfaces $xyz = c$. On the other hand, since $xzdx + yzdy + xydz = 0$ does not satisfy the condition for integrability, there is no family of surfaces which is cut orthogonally by the curves whose equations are $x - c_1y = 0$, $xy - z^2 = c_2$ (Ex. 2, §XI·3). The converse problem of finding the orthogonal trajectories of a family of surfaces whose equation is $f(x, y, z) = c$ requires the solution of the system of equations

---

* From now on we shall use the three letters x, y, z, instead of x, y, t.

$$\frac{dx}{\dfrac{\partial f}{\partial x}} = \frac{dy}{\dfrac{\partial f}{\partial y}} = \frac{dz}{\dfrac{\partial f}{\partial z}}.$$

In general, these can be solved.

Exercise **1.** Find the orthogonal trajectories of the family of spheres $x^2 + y^2 + z^2 = c$.

2. Find the orthogonal trajectories of the family of spheres $x^2 + y^2 + z^2 = cx$.

3. Find the orthogonal trajectories of the family of central quadrics $hx^2 + ky^2 + lz^2 = c$.

4. Find the orthogonal trajectories of the family of hyperbolic paraboloids $xy = cz$.

5. Find the orthogonal trajectories of the family of elliptic paraboloids $x^2 + y^2 + cz = 0$.

6. Find the orthogonal trajectories of the family of circular cones $x^2 + y^2 + cz^2 = 0$.

7. Find the orthogonal trajectories of the family of hyperbolic paraboloids $yz + zx + xy = c$.

XI·5. System of total differential equations. It can be shown that the general solution of two total differential equations in three variables,

$$\text{(1)} \qquad \begin{cases} P_1dx + Q_1dy + R_1dz = 0, \\ P_2dx + Q_2dy + R_2dz = 0, \end{cases}$$

consists of two relations among the variables involving two arbitrary constants. In actual practice, we proceed as follows:

If each of equations (1) separately satisfies the condition for integrability [§X·1, (3)], we solve each one, and, thus, obtain the solution of our system.

If only one of the equations satisfies the condition for integrability, we integrate it and obtain a relation $\phi(x, y, z) = c_1$. Since the other equation in (1) is non-integrable, we employ the method of §X·6, and use $\phi(x, y, z) = c_1$ as the assumed relation. The resulting solution of the non-integrable equation is the solution of the system (1).

If neither of the equations is separately integrable, it is sometimes desirable to put (1) in the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \tag{2}$$

where $P = Q_1R_2 - Q_2R_1$, $Q = R_1P_2 - R_2P_1$, $R = P_1Q_2 - P_2Q_1$. The methods of §XI·3 may now be tried.

If they are not applicable, we select one of the variables as the independent one, e.g. z, and write the equations in the form

$$\begin{cases} \dfrac{dx}{dz} = \dfrac{P}{R}, \\[2ex] \dfrac{dy}{dz} = \dfrac{Q}{R}. \end{cases} \tag{3}$$

The general method of §XI·1 may now be employed.

XI·6. Differential equations of higher order than the first reducible to systems of equations of the first order. Given a single equation with one dependent variable, we shall suppose it solved for the highest ordered derivative. Thus, consider

$$\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\right). \tag{1}$$

If we introduce the new variables

$$y_1 = \frac{dy}{dx}, \quad y_2 = \frac{d^2y}{dx^2}, \cdots, y_{n-1} = \frac{d^{n-1}y}{dx^{n-1}},$$

as we did in §IX·1, the equation (1) may be replaced by the system of n equations of the first order

$$(2)\qquad \begin{cases} \dfrac{dy}{dx} = y_1, \\ \dfrac{dy_1}{dx} = y_2, \\ \cdots\cdots\cdots\cdots\cdots \\ \dfrac{dy_{n-1}}{dx} = f(x, y, y_1, \cdots, y_{n-1}). \end{cases}$$

As an illustration, the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

is equivalent to the system

$$\frac{dy}{dx} = y_1, \quad \frac{dy_1}{dx} = -y,$$

or

$$\frac{dx}{1} = \frac{dy}{y_1} = \frac{dy_1}{-y}.$$

Using the last two terms, we have

$$y^2 + y_1^2 = c_1^2, \quad \text{or} \quad y_1 = \sqrt{c_1^2 - y^2}.$$

$$\therefore \frac{dy}{\sqrt{c_1^2 - y^2}} = dx.$$

Integrating, we have

$$\sin^{-1}\frac{y}{c_1} = x + c_2, \quad \text{or} \quad y = c_1 \sin (x + c_2).$$

In an analogous manner, a system of n equations of any order in n dependent variables may be replaced by a system of equations of the first order, by taking as new variables all the derivatives of the dependent variables except those of highest order. Thus, by letting $x_1 = dx/dt$, the system of equations of §XI·2 may be written

$$\begin{cases} \dfrac{dx}{dt} = x_1, \\ \dfrac{dy}{dt} = x_1 + x - \cos t, \\ \dfrac{dx_1}{dt} = x_1 - 2x + y + e^{2t} - \cos t. \end{cases}$$

XI·7. Summary. To solve a system of two ordinary differential equations involving two dependent variables, we differentiate these equations a sufficient number of times to enable us to eliminate one of the dependent variables and all of its derivatives, thus giving rise to a single equation involving only one dependent variable and its derivatives. We integrate this, and, substituting for this variable and its derivatives their values in terms of the independent variable in either of the original equations, we obtain a new differential equation, whose solution determines the other dependent variable. Or, we can treat both of the dependent variables symmetrically by solving for each one separately. The relations among the constants of integration are found by substituting in one of the original equations, or, if necessary, in both of them (§XI·1).

While this method is frequently not practicable, it can be carried out very readily in case the equations are linear with constant coefficients (§XI·2).

If the equations are of the first order, special methods can, at times, be employed (§XI·3).

A system of two total differential equations in three variables can be written as a system of ordinary differential equations, to which the methods of §XI·1 and §XI·3 may be applicable (§XI·5).

A single differential equation in one dependent variable of higher order than the first may be replaced by a corresponding system of differential equations of the first order, to which at times the special methods of §XI·3 apply (§XI·6).

It is almost needless to add that, if each of a system of equations involves a single dependent variable, each is to be integrated independently of the others. Thus, see Ex. 1, 2, 3, and 4 below.

Exercise **1.** Find the path traced by a particle, moving in a vacuum and acted upon by gravity only, if it is given an initial velocity v_0 in a direction making an angle α with the horizontal plane.

2. If the particle in Ex. 1 moves in a medium which exerts a resisting force proportional to the velocity, find its path.

3. A particle moves subject to an attracting force which varies as the distance of the particle from a fixed point O, called the *center of force,* and is directed along the line joining the particle with O. Taking O as the origin of coördinates, determine the motion of the particle, if it starts to move from a point on the axis of x at a distance a from the center, with an initial velocity v_0 making an angle α with the axis of x.

[If the acceleration due to the attracting force is P, and r is the distance of the particle from the center of the force, the equations of motion are

$$\frac{d^2x}{dt^2} = -P\frac{x}{r}, \quad \frac{d^2y}{dt^2} = -P\frac{y}{r}.$$

In this case $P = k^2r$.]

4. If the force is a repulsive one, study the motion of the particle in Ex. 3.

5. A solid of revolution, with one point of its axis of symmetry fixed, is acted upon by gravity only. Find its angular velocity and the position of the instantaneous axis of rotation in the body.

[If A, B, C are the moments of inertia of the body with respect to the principal axes of the momental ellipsoid about the fixed point, and p, q, r are the components of the angular velocity on those axes at any instant, Euler's equations are

$$\begin{aligned} A\frac{dp}{dt} + (C - A)qr &= 0, \\ A\frac{dq}{dt} - (C - A)rp &= 0, \\ C\frac{dr}{dt} \qquad\qquad &= 0, \end{aligned}$$

since $B = A$.]

6. The component of the velocity of a particle parallel to each of the coördinate axes is proportional to the product of the other two coördinates. Find its path. Also find the time of describing a given portion in case the curve passes through the origin.

7. A charged particle starts from rest in the xy-plane at a time $t = 0$ and is acted upon by a uniform electric field of strength X, parallel to the x-axis, and by a uniform magnetic force H, parallel to the z-axis. Find the path of the particle, assuming the equations of motion to be

$$m\frac{d^2x}{dt^2} = eX - eH\frac{dy}{dt},$$

$$m\frac{d^2y}{dt^2} = eH\frac{dx}{dt}.$$

Here m, e, X, H are constants. [H. Bateman, *Differential Equations.*]

CHAPTER XII

PARTIAL DIFFERENTIAL EQUATIONS

XII·1. Primitive involving arbitrary constants. Partial differential equations may be obtained from primitives involving either arbitrary constants or arbitrary functions. In this section, we shall restrict ourselves to the first of these cases.

Consider the family of spheres of fixed radius R, with their centers lying in the plane $z = 0$. The equation of this family is obviously

$$(x - a)^2 + (y - b)^2 + z^2 = R^2, \tag{1}$$

where a and b are arbitrary constants.

We shall consider x and y as independent variables, and shall put

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial^2 z}{\partial x^2} = r,\ \frac{\partial^2 z}{\partial x \partial y} = s,\ \frac{\partial^2 z}{\partial y^2} = t.$$

Differentiating (1) with respect to x and y respectively, we obtain

$$x - a + zp = 0, \tag{2}$$

$$y - b + zq = 0. \tag{3}$$

Eliminating a and b from (1), (2), (3), we have

$$z^2(p^2 + q^2 + 1) = R^2, \tag{4}$$

which is known as the *differential equation* corresponding to the *primitive* (1); on the other hand, (1) is said to be a *solution* of (4).

This suggests the general mode of procedure. Starting with a relation involving two arbitrary constants and three variables,

$$\phi(x, y, z, a, b) = 0, \tag{1'}$$

where z is taken as the dependent variable, we obtain, on differentiating with respect to x and y respectively,

$$\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} = 0, \tag{2'}$$

$$\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} = 0. \tag{3'}$$

We now have three equations from which, in general, a and b can be eliminated. When this is done, there results

$$f(x, y, z, p, q) = 0, \tag{4'}$$

a differential equation of the first order which has (1′) for its primitive.

If the primitive involves more than two independent arbitrary constants, more than three equations will be necessary to eliminate them. Hence, if only two independent variables are involved, the resulting differential equation will be of higher order than the first. Thus, consider

$$ax^2 + by^2 + cz^2 = 1. \tag{5}$$

Differentiating with respect to x and y respectively, we have

$$ax + czp = 0, \tag{6}$$

$$by + czq = 0. \tag{7}$$

We have now only three equations from which to eliminate a, b, c. Hence we must differentiate again. Differentiating (6) with respect to x, we have

$$a + c(zr + p^2) = 0. \tag{8}$$

Eliminating a, b, c from (5), (6), (7), (8), we have

$$\begin{vmatrix} x^2 & y^2 & z^2 & -1 \\ x & 0 & zp & 0 \\ 0 & y & zq & 0 \\ 1 & 0 & zr + p^2 & 0 \end{vmatrix} = 0, \quad \text{or}$$

$$xzr + xp^2 - zp = 0, \tag{9}$$

which is a differential equation resulting from (5) as a primitive.

If we differentiate (6) with respect to y, we obtain

$$czs + cpq = 0; \tag{8'}$$

whence, on eliminating a, b, c from (5), (6), (7), (8′), we have

$$zs + pq = 0, \tag{9'}$$

which is also a differential equation resulting from (5) as a primitive. Again, differentiating (7) with respect to y, we obtain

$$b + c(zt + q^2) = 0. \tag{8''}$$

Eliminating a, b, c from (5), (6), (7), (8″), we have

$$yzt + yq^2 - zq = 0, \tag{9''}$$

which is also a differential equation resulting from (5) as a primitive.

Since (9), (9′), (9″) are all of the second order, there is no choice among them, and all may be said to be differential equations belonging to (5).

There is no such ambiguity in the case of the partial differential equation of the first order arising from a primitive involving two arbitrary constants.

A primitive involving only one arbitrary constant gives rise to two independent differential equations of the first order.

As we shall see in the next chapter, a solution of a partial differential equation of the first order which contains two independent arbitrary constants plays a very important rôle in determining all solutions of the differential equation.

In order to obtain a unique partial differential equation of the second order, the primitive must contain five independent arbitrary constants. For, in this case, we obtain two additional relations by differentiating once with respect to each of the independent variables, and three additional relations involving second partial derivatives by differentiating the primitive twice with respect to x, once each with respect to x and y, and twice with respect to y.

In the general case, it can be shown that, in order that a primitive may give rise to a unique partial differential equation of the nth order, it must contain $n(n + 3)/2$ independent arbitrary constants. This number is the sum of all the integers from 2 to $n + 1$.

Form the differential equations having the following equations for primitives, a, b, c, f, g being the arbitrary constants to be eliminated:

***Exercise* 1.** $(x - a)^2 + (y - a)^2 + z^2 = b^2$.

2. $(x - a)^2 + (y - b)^2 + z^2 = a^2$. **3.** $z = ax + by + \sqrt{a^2 + b^2}$.

4. $z = (x + a)(y + b)$.

5. $(x - a)^2 + (y - b)^2 + (z - c)^2 = f$.

6. $z = ax + by + cxy$. **7.** $ax^2 + by^2 + cz^2 + 2gx + 2fy = 1$.

XII·2. Primitives involving arbitrary functions. In this section, we shall briefly indicate how partial differential equations may arise from primitives involving arbitrary functions. In doing this, we shall restrict our consideration to two simple types of primitives.

1° Let u and v be two known functions of x, y, z. We shall start with the relation

$$\phi(u, v) = 0 \tag{1}$$

as a primitive, where $\phi(u, v)$ is an arbitrary function of u and v. Differentiating with respect to x and y respectively, we have

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0, \tag{2}$$

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0. \tag{3}$$

Here ϕ does not appear in (2) or (3). Moreover, (2) and (3) are linear and homogeneous in $\partial\phi/\partial u$ and $\partial\phi/\partial v$. Hence these can be eliminated by equating to zero the determinant of the coefficients; thus

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z} p & \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} q \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z} p & \dfrac{\partial v}{\partial y} + \dfrac{\partial v}{\partial z} q \end{vmatrix} = 0,$$

or

$$Pp + Qq = R, \tag{4}$$

where

$$P = \begin{vmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial y} \end{vmatrix}, \quad Q = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial z} \end{vmatrix}, \quad R = \begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial x} \end{vmatrix}.$$

Equation (4) is the partial differential equation corresponding to the primitive (1). It is of a special form and is known as a *linear partial differential equation of the first order.* By such a partial differential equation we mean one that is linear in the derivatives of the dependent variable (the way in which the dependent variable itself enters playing no rôle).*

2° If u and v are two known functions of x, y, z, and we have

$$f[x, y, z, \phi(u), \psi(v)] = 0, \tag{5}$$

where f is some known function, but ϕ and ψ are arbitrary functions, we obtain, on differentiating,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial \phi} \phi'(u) \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p \right) + \frac{\partial f}{\partial \psi} \psi'(v) \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \right) = 0, \tag{6}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial \phi} \phi'(u) \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q \right) + \frac{\partial f}{\partial \psi} \psi'(v) \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \right) = 0. \tag{7}$$

Since (6) and (7) introduce two new functions $\phi'(u)$, $\psi'(v)$, five equations are necessary to eliminate the four functions $\phi(u)$, $\psi(v)$, $\phi'(u)$, $\psi'(v)$. If we differentiate (6) and (7), we obtain three new equations involving second derivatives of z and the new functions $\phi''(u)$ and $\psi''(v)$. In all we have now six equations from which, in general, it is not possible to eliminate the six arbitrary functions. Hence, we must differentiate again, this time obtaining four new equations, involving third derivatives of z and the two new functions $\phi'''(u)$ and $\psi'''(v)$. That is, we have now ten equations from which the eight functions can be eliminated, in general, in two distinct ways. Hence we are led in the general case to two differential equations of the third order.

---

* The student will note the difference between this definition and that given for a linear ordinary differential equation of the first order (§II·9).

In the case of special forms of the function f, we can eliminate the six functions $\phi(u)$, $\phi'(u)$, $\phi''(u)$, $\psi(v)$, $\psi'(v)$, $\psi''(v)$ from the first six equations arising in the above process. Thus, for example, suppose $f \equiv w - [\phi(u) + \psi(v)] = 0$, where w is a known function of x, y, z. Then

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial z} p - \left[\phi'(u)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \psi'(v)\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right)\right] = 0,$$

$$\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} q - \left[\phi'(u)\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \psi'(v)\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right)\right] = 0.$$

These two equations contain $\phi'(u)$ and $\psi'(v)$, but do not involve $\phi(u)$ and $\psi(v)$. Hence, since the three equations obtained by differentiating them involve only $\phi'(u)$, $\phi''(u)$, $\psi'(v)$, $\psi''(v)$, these four functions can, in general, be eliminated from the five equations in which they enter. A single differential equation of the second order arises which has $w = \phi(u) + \psi(v)$ for primitive involving the two arbitrary functions ϕ and ψ.*

Find the differential equations arising from the following primitives:

***Exercise* 1.** $\phi(x + y + z, x^2 + y^2 + z^2) = 0$.

Letting $x + y + z = u$, $x^2 + y^2 + z^2 = v$, we have $\phi(u, v) = 0$.

---

* The five equations are linear in $\phi'(u)$, $\phi''(u)$, $\psi'(v)$, $\psi''(v)$. Hence the elimination can be effected readily. Moreover, r, s, t enter linearly also, and in such a way that the resulting differential equation is also linear in them, that is, it has the form $Rr + Ss + Tt = V$, where R, S, T, V are functions of x, y, z, p, q. Such linear partial differential equations of the second order yield more readily to general methods of solution than do non-linear partial differential equations of the second order.

Differentiating, we obtain

$$\frac{\partial\phi}{\partial u}(1+p)+\frac{\partial\phi}{\partial v}(2x+2zp)=0,$$

$$\frac{\partial\phi}{\partial u}(1+q)+\frac{\partial\phi}{\partial v}(2y+2zq)=0.$$

$$\therefore\ (1+p)(y+zq)-(1+q)(x+zp)=0,$$

or

$$(y-z)p+(z-x)q=x-y.$$

2. $\phi\left(z^2-xy,\dfrac{y}{x}\right)=0.$ **3.** $\phi(x^2+y^2, z-xy)=0.$

4. $z=\phi(x+y)+\psi(x-y).$ **5.** $z=\phi(x+y)+\psi(xy).$

XII·3. Solution of a partial differential equation. Since a primitive which gives rise to a differential equation is obviously a solution of that equation, we see that arbitrary functions as well as arbitrary constants may enter into the solutions of partial differential equations. From the general existence theorems for partial differential equations, it is seen that, in general, a partial differential equation, or system of such equations, has a solution containing a definite number of arbitrary functions. As an arbitrary function may contain an indefinite number of arbitrary constants, a solution involving an arbitrary function is much more general than one that contains any fixed number of arbitrary constants. We shall speak of the solution given by the existence theorem (which is a solution involving one or more arbitrary functions) as *the general solution.*

The mere statement of the existence theorem for a system of partial differential equations of any order is quite complicated. We shall give here only a statement in the cases of a single equation of the first and second orders in three variables:

1° Consider the equation $f(x, y, z, p, q)=0$. We shall suppose that p actually appears.* When solved for p, the equation takes the form

$$p=F(x, y, z, q).$$

---

* If p is absent, then q must appear, and the argument here employed can be carried out by interchanging in it p and q, and x and y.

The existence theorem states that, *if* $F(x, y, z, q)$ *is regular in the regions of* $x = x_0$, $y = y_0$, $z = z_0$, $q = q_0$, *and if* $\phi(y)$ *is any arbitrarily chosen function of* y, *regular in the region of* y_0 *and such that* $\phi(y_0) = z_0$, $\phi'(y_0) = q_0$, *there exists one and only one solution,* $z = \psi(x, y)$, *which is regular in the regions of* $x = x_0$, $y = y_0$, *and which reduces to* $z = \phi(y)$ *when* $x = x_0$.

Geometrically this means that, in general, *given any curve* $z = \phi(y)$ *in the plane* $x = x_0$, *there exists one and only one integral surface (in any region for which there are no singular points* of the differential equation) passing through that curve.*

This can be generalized. It can be shown that, in general, a unique integral surface can be found passing through any arbitrarily chosen curve, whether plane or skew (as long as we avoid singular points of the equation). See Goursat-Bourlet, *Équations aux dérivées partielles du premier ordre*, p. 21.

2° Consider the equation $f(x, y, z, p, q, r, s, t) = 0$. If this contains neither r nor t, a linear transformation will introduce one or both of these. We shall suppose then that one of them appears; and there will be no loss in supposing that it is r. When solved for r, the equation takes the form

$$r = F(x, y, z, p, q, s, t).$$

The existence theorem states that, *if* F *is regular in the regions of* $x = x_0$, $y = y_0$, $z = z_0$, $p = p_0$, $q = q_0$, $s = s_0$, $t = t_0$, *and if* $\phi(y)$ *and* $\psi(y)$ *are any arbitrarily chosen functions of* y, *regular in the region of* y_0 *and such that* $\phi(y_0) = z_0$, $\phi'(y_0) = q_0$, $\phi''(y_0) = t_0$, $\psi(y_0) = p_0$, $\psi'(y_0) = s_0$, *there exists one and only one solution,* $z = \Phi(x, y)$, *which is regular in the regions of* $x = x_0$, $y = y_0$, *and such that* $z = \phi(y)$ *and* $p = \psi(y)$ *when* $x = x_0$.

Interpreted geometrically, this means that, in general, *given any curve* $z = \phi(y)$ *in the plane* $x = x_0$; *there exists an indefinite number of integral surfaces which pass through it. But, if at each point of this curve we fix a line, there exists one and only one integral surface through the curve and normal to these lines.* For, the direction cosines of the normal to the integral surface are proportional to p, q, -1. When the curve is given, we know $\phi(y)$. The q at each point of this curve is $\phi'(y)$, which is also known. Hence the assigning of a normal at each point determines p, which is our $\psi(y)$. Once $\phi(y)$ and $\psi(y)$ are given, the existence theorem assures a unique integral surface. As in the previous case, no singular points of the differential equation may appear in the regions in which we are interested.

This theorem also may be generalized to apply to any curve whether plane or skew.

---

* By a *singular point* of the equation, we mean one whose coördinates together with the corresponding value of q determine a set of values in the regions of which F ceases to be regular.

CHAPTER XIII

PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

XIII·1. Linear partial differential equation of the first order. Method of Lagrange. Lagrange deduced the following very elegant method of solving a linear partial differential equation of the first order. The general type of such an equation for one dependent and two independent variables is

$$Pp + Qq = R, \tag{1}$$

where P, Q, R are functions of x, y, z.

Consider the linear equation with three independent variables x, y, z

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = 0. \tag{2}$$

It is *homogeneous* (in the partial derivatives), and the coefficients are functions of the independent variables only. If $u(x, y, z) = c$ satisfies (1), $f = u(x, y, z)$ will be a solution of (2); for, differentiating $u = c$, first with respect to x and then with respect to y and solving the results for p and q, we find

$$p = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial z}}, \qquad q = -\frac{\dfrac{\partial u}{\partial y}}{\dfrac{\partial u}{\partial z}}.$$

Substituting these in (1), we see that $f = u$ satisfies (2).

Conversely, if $f = u$ is a solution of (2), $u = c$ will satisfy (1). For, from (2) we have, on putting $f = u$ and solving for R,

$$-\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial z}}P - \frac{\dfrac{\partial u}{\partial y}}{\dfrac{\partial u}{\partial z}}Q = R.$$

But

$$-\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial z}} = p, \quad -\frac{\dfrac{\partial u}{\partial y}}{\dfrac{\partial u}{\partial z}} = q$$

when $u = c$. Hence (1) follows. From this we see that the problem of solving (1) is equivalent to that of solving (2).

Consider now the system of ordinary differential equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{3}$$

If $f = u(x, y, z)$ and $f = v(x, y, z)$ are two independent solutions of (2), $u = c_1$ and $v = c_2$ constitute a solution of (3); for, if we multiply numerator and denominator of these fractions by $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$ respectively, it follows, by composition, that each of the fractions of (3) is equal to

$$\frac{\dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy + \dfrac{\partial u}{\partial z}dz}{P\dfrac{\partial u}{\partial x} + Q\dfrac{\partial u}{\partial y} + R\dfrac{\partial u}{\partial z}}.$$

Since, by hypothesis, the denominator is zero, the numerator (which is du) must also be zero. Hence, $u = c_1$ may be used as one of the two relations constituting a solution of (3). Similarly, $v = c_2$ can be shown to be another relation of this sort. Conversely, if $u = c_1$, $v = c_2$ is a solution of (3), each of $f = u$ and $f = v$ satisfies (2). For, by differentiation, we have

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0, \quad \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0. \tag{4}$$

Moreover, since $u = c_1$, $v = c_2$ is a solution of (3), it follows that

$$dx : dy : dz = P : Q : R.$$

Replacing the former by the latter in each of (4), we have

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0, \quad P\frac{\partial v}{\partial x} + Q\frac{\partial v}{\partial y} + R\frac{\partial v}{\partial z} = 0;$$

i.e., (2) is satisfied by $f = u$ and by $f = v$.

We have thus shown that the problem of solving the linear partial differential equation (1) may be replaced by that of solving the system of ordinary equations (3).

Moreover, if $u(x, y, z,) = c_1$, $v(x, y, z) = c_2$ is the general solution of (3), and if $\phi(u, v)$ is an arbitrarily selected function of u and v, $f = \phi(u, v)$ will satisfy (2). For, substituting it in the left-hand member of (2), we have

$$\frac{\partial \phi}{\partial u}\left(P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(P\frac{\partial v}{\partial x} + Q\frac{\partial v}{\partial y} + R\frac{\partial v}{\partial z}\right).$$

This vanishes, since both u and v satisfy (2). Hence $f = \phi(u, v)$ is also a solution of (2).

Therefore, $\phi(u, v) = 0$* is the general solution of (1), since it contains an arbitrary function. Moreover, since ϕ is an arbitrary function, there is no loss of generality in replacing the arbitrary constant in the right-hand member by zero.

This method of solution may be extended, with only obvious modifications, to a linear equation with n independent variables. Accordingly, we shall merely state the result:

To find the general solution of

$$P_1\frac{\partial z}{\partial x_1} + P_2\frac{\partial z}{\partial x_2} + \cdots + P_n\frac{\partial z}{\partial x_n} = R,$$

solve

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R}.$$

If the general solution of this system is

$$u_1 = c_1,\ u_2 = c_2, \cdots, u_n = c_n,$$

then $\phi(u_1, u_2, \cdots, u_n) = 0$, *where* ϕ *is an arbitrary function of* $u_1, u_2, \cdots, u_n$, *will be the general solution of* (1).

Exercise 1. $xzp + yzq = xy$.

To solve this, we consider the system of equations (3)

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}.$$

---

* The solution may also be written in the form $u = f(v)$, or $v = \psi(u)$, where f and ψ are arbitrary functions.

Multiplying numerator and denominator of the three members by y, x, $-2z$ respectively, we find that

$$ydx + xdy - 2zdz = 0. \quad \text{(Method 3° (}c\text{), §XI·3.)}$$

$$\therefore xy - z^2 = c_1.$$

Besides, $\dfrac{dx}{xz} = \dfrac{dy}{yz}$, or $\dfrac{dx}{x} = \dfrac{dy}{y}$, has the solution $\dfrac{y}{x} = c_2$.

$$\therefore \phi\left(xy - z^2, \frac{y}{x}\right) = 0 \text{ is the general solution.}$$

***Exercise* 2.** $-y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} + (1 + z^2)\dfrac{\partial u}{\partial z} = 0.$

Consider the system of ordinary differential equations

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{1 + z^2} = \frac{du}{0}.$$

We have at once

$$u = c_1.$$

The differential equation obtained by using the first two members has the solution

$$x^2 + y^2 = c_2.$$

Multiplying numerator and denominator of the first two members by $-y$ and x respectively, we have, by composition,

$$\frac{xdy - ydx}{x^2 + y^2} = \frac{dz}{1 + z^2}. \quad \text{(Method 3° (}a\text{), §XI·3.)}$$

$$\therefore \tan^{-1}\frac{y}{x} - \tan^{-1} z = c,$$

or

$$\frac{y - xz}{x + yz} = c_3.$$

$\therefore \phi\left(u, x^2 + y^2, \dfrac{y - xz}{x + yz}\right) = 0$, or $u = f\left(x^2 + y^2, \dfrac{y - xz}{x + yz}\right)$, is the general

3. $yp - xq = x^2 - y^2$. **4.** $(y - z)p + (z - x)q = x - y$.

5. $(1 + y)p + (1 + x)q = z$.

6. $(xz + yz)p + (xz - yz)q = x^2 + y^2$.

XIII·2. Equation in which p or q is absent. If either of the first derivatives is absent, the differential equation may be solved for the derivative that appears, thus giving rise to a special form of linear partial differential equation. If, for example, q is absent, so that the equation is of the form

$$f(x, y, z, p) = 0, \tag{1}$$

we have, on solving for p,

$$p = F(x, y, z). \tag{2}$$

The corresponding system of ordinary differential equations (3), §XIII·1, is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dz}{F(x, y, z)}. \tag{3}$$

Here one relation is $y = a$. The other is obtained as a solution of

$$\frac{dz}{dx} = F(x, a, z). \tag{4}$$

If this solution is $\phi(x, a, z) = c$, the general solution of the original partial differential equation (1) is

$$\phi(x, y, z) = \psi(y), \tag{5}$$

where $\psi(y)$ is an arbitrary function of y.

As a matter of fact, it is not necessary to solve equation (1) for p and thus reduce it to a linear equation. Writing (1) in the form

$$f\left(x, y, z, \frac{dz}{dx}\right) = 0, \tag{6}$$

where dz/dx is an ordinary derivative and y is considered as constant, we solve this ordinary differential equation by any of the methods of Chapters II and IV. If its solution is

$$\phi(x, y, z, c) = 0,$$

where y is still a constant, the general solution of (1) will be

$$\phi[x, y, z, \psi(y)] = 0, \tag{7}$$

where y is now variable and $\psi(y)$ is an arbitrary function.

If p is absent, we consider x as constant and q as the ordinary derivative dz/dy, solve the resulting ordinary differential equation, and in its solution replace the arbitrary constant by an arbitrary function of x.

***Exercise* 1.** $xp^2 - 2zp + xy = 0.$

Considering y as constant, we write $p = dz/dx$, thus obtaining an ordinary differential equation of the first order. We shall use the method of §IV·2. Writing z' for dz/dx and solving for z, we have

$$2z = xz' + \frac{xy}{z'}. \tag{8}$$

Differentiating with respect to x and arranging the terms, we have

$$\left(z' - \frac{y}{z'}\right)\left(1 - \frac{x}{z'}\frac{dz'}{dx}\right) = 0.$$

Neglecting the factor $z' - y/z'$ and integrating the resulting equation, we find

$$z' = cx.$$

Eliminating z' between this and equation (8), we find the solution of the ordinary differential equation (8) in the form

$$2z = cx^2 + \frac{y}{c};$$

the solution of the original partial differential equation is, therefore,

$$2z = x^2\psi(y) + \frac{y}{\psi(y)}. \tag{9}$$

The singular solution of (8), viz.

$$z^2 - x^2y = 0,$$

is also a solution of the partial differential equation. Since it cannot be obtained from the general solution (9) by any form of the arbitrary function $\psi(y)$, it is known as a *special solution.**

2. $(x - y)q^2 - (x + z) = 0.$ **3.** $x^2 + (z - yq)^2 = 1.$

4. $y^2(p^2 - 1) = x^2p^2.$ **5.** $q^2 + (2x + y)q + x^2 - z = 0.$

REMARK. *Special solutions* may also arise in the case of linear partial differential equations of the first order, as treated in §XIII·1, when the coefficients P, Q, R are not single-valued.* Thus, consider

---

* This term is employed by Professor Hill, in the memoir cited on the next page.

6. $(1 + \sqrt{z - x - y})p + q = 2.$

To solve this, we consider the system of ordinary differential equations

$$\frac{dx}{1 + \sqrt{z - x - y}} = \frac{dy}{1} = \frac{dz}{2}.$$

Using the method 3° (*a*) of §XI·3, we have, on putting $\lambda = -1$, $\mu = -1, \nu = 1$,

$$\frac{dz - dx - dy}{-\sqrt{z - x - y}} = \frac{dy}{1}.$$

Integrating, we find

$$2\sqrt{z - x - y} + y = a.$$

A second relation is evidently

$$2y - z = b.$$

Hence the general solution is

$$\phi(2y - z, \quad 2\sqrt{z - x - y} + y) = 0.$$

But $z - x - y = 0$ is evidently also a solution; for, then $p = 1$ and $q = 1$, and the differential equation is satisfied. Moreover, this special solution cannot be obtained from the general solution by any choice of the arbitrary function ϕ.

As in many cases of this kind, the existence of $z - x - y = 0$ as a special solution may be shown by replacing the radical appearing in the differential equation by a new variable. Thus, let

$$\sqrt{z - x - y} = w, \quad \text{or} \quad z - x - y = w^2.$$

Then

$$p = 2w\frac{\partial w}{\partial x} + 1 \quad \text{and} \quad q = 2w\frac{\partial w}{\partial y} + 1.$$

Putting these in the differential equation, we find

$$w\left[2(1 + w)\frac{\partial w}{\partial x} + 2\frac{\partial w}{\partial y} + 1\right] = 0.$$

Equating the factor w to zero, we obtain the special solution.

---

* For a detailed study of these and other special solutions, see M.T.M. Hill, *Proceedings of the London Mathematical Society*, Series 2, Vol. 16 (1917), pp. 219–272. Some of the exercises of this section are taken from this memoir.

As an exercise the student should show that the special solution in the case of Ex. 1 becomes apparent by making the change of dependent variable

$$\sqrt{z^2 - x^2y} = w,$$

this radical appearing when the differential equation is solved (algebraically) for p.

7. $p + xq = xy\sqrt{z}$. **8.** $\sqrt{x}\,p + \sqrt{y}\,q = \sqrt{z}$.
9. $(1 + x^2)p + (1 + y^2)q = \sqrt{1 - z^2}$.
10. $[1 + (z - x - y)\log(z - x - y)]p + q = 2$.

XIII·3. Integrating factor for the ordinary differential equation $Mdx + Ndy = 0$. We are now in a position to treat, in a general manner, the problem of finding an integrating factor for an ordinary differential equation of the first order and degree. We have seen (§II·7) that the necessary and sufficient condition that μ be an integrating factor of $Mdx + Ndy = 0$ is

$$\frac{\partial(\mu N)}{\partial x} - \frac{\partial(\mu M)}{\partial y} = 0,$$

or

$$\mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) + N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = 0;$$

whence

$$\frac{N}{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}\frac{\partial \mu}{\partial x} + \frac{M}{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}\frac{\partial \mu}{\partial y} = \mu.^* \tag{1}$$

In order to solve (1), we consider the system of ordinary differential equations

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}\,dx = \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}\,dy = \frac{d\mu}{\mu}. \tag{2}$$

In actual practice, when trying to integrate $Mdx + Ndy = 0$, we are not desirous of finding the most general form for μ; as a matter of fact, a simple form is usually preferable. It should be noted, however, that the method of this section is not usually practical. But, for special forms of M and N, a solution of (2) can be found. Thus, the following general classes of equations for which we can find a solution of (2) may be noted here.*

---

* Since the number of solutions of this linear partial differential equation is infinite, we see again that an ordinary differential equation of the first order has an infinite number of integrating factors (§II·6).

1° If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ is a function of x only, e.g. $f_1(x)$, we find, from (2), that $\mu = e^{\int f_1(x)dx}$ is an integrating factor (§II·13).

2° If $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$ is a function of y only, e.g. $f_2(y)$, then $\mu = e^{\int f_2(y)dy}$ is obviously an integrating factor (§II·13).

3° If the equation is linear, then $M = Py - Q$, $N = 1$, and equations (2) become $Pdx = \dfrac{-P}{Py-Q}dy = \dfrac{d\mu}{\mu}$. Hence, $\mu = e^{\int Pdx}$ is an integrating factor (§II·9).

4° If M and N are homogeneous and of the same degree n, we obtain, by composition, after we have multiplied numerator and denominator of the first two members of (2) by y and x respectively,

$$\frac{\left(y\dfrac{\partial M}{\partial y}-y\dfrac{\partial N}{\partial x}\right)dx+\left(x\dfrac{\partial N}{\partial x}-x\dfrac{\partial M}{\partial y}\right)dy}{xM+yN}=\frac{d\mu}{\mu}. \tag{3}$$

By Euler's theorem for homogeneous functions, we have

$$x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y}=nM \quad \therefore y\frac{\partial M}{\partial y}=nM-x\frac{\partial M}{\partial x},$$

$$x\frac{\partial N}{\partial x}+y\frac{\partial N}{\partial y}=nN \quad \therefore x\frac{\partial N}{\partial x}=nN-y\frac{\partial N}{\partial y};$$

whence (3) may be written

$$\frac{x\left(\dfrac{\partial M}{\partial x}dx+\dfrac{\partial M}{\partial y}dy\right)+y\left(\dfrac{\partial N}{\partial x}dx+\dfrac{\partial N}{\partial y}dy\right)-n(Mdx+Ndy)}{xM+yN}=-\frac{d\mu}{\mu}.$$

---

* These were enumerated in Chapter II among the equations for which integrating factors can be found.

But $Mdx + Ndy = 0$. Hence we may add $(n + 1)\ (Mdx + Ndy)$ to the numerator on the left without altering its value. Doing so, we have

$$\frac{d(xM + yN)}{xM + yN} = -\frac{d\mu}{\mu}.$$

Integrating, we have

$$\mu = \frac{1}{xM + yN}\ (\text{§II·11}).$$

XIII·4. Geometrical considerations. It is well known that the direction cosines of the normal to the surface $z = \phi(x, y)$ at any point (x, y, z) on the surface are proportional to

$$p : q : -1.$$

A partial differential equation of the first order

(1) $$f(x, y, z, p, q) = 0$$

determines a relation between p and q at each point (x, y, z) in space. The aggregate of all the lines through a point (x, y, z), whose direction cosines are proportional to

$$p : q : -1,$$

where p and q satisfy the relation (1), forms, in general, a cone with its vertex at (x, y, z), the various lines being elements of the cone.

If the normal to the surface $z = \phi(x, y)$ at each point (x, y, z) coincides with a generator of the cone determined by the differential equation at that point, the equation of the surface will evidently be a solution of the differential equation. For, the values of p and q at each point (x, y, z) on the surface, determined by differentiating the equation of the surface, will satisfy the differential equation.

In the case of a linear partial differential equation

(2) $$Pp + Qq - R = 0,$$

the cone referred to above becomes a plane. For, the functions P, Q, R assume definite values at any given point (x, y, z),* which may be taken as proportional to the direction cosines of a line through the point (x, y, z). The differential equation (2) asserts that this line is perpendicular to all the lines through (x, y, z) whose direction cosines are proportional to

---

* The points under consideration are restricted to a region in which P, Q, R are finite, single-valued, and do not vanish simultaneously.

$$p : q : -1,$$

i.e., to all the elements of the cone. Hence the cone is a plane. If $z = \phi(x, y)$ is an integral surface passing through (x, y, z), its normal at (x, y, z) must lie in this plane.

XIII·5. Non-linear partial differential equation of the first order. Singular solution. We have seen (§XII·1) that a primitive

$$\phi(x, y, z, a, b) = 0, \tag{1}$$

which involves two arbitrary constants, gives rise to a unique partial differential equation of the first order

$$f(x, y, z, p, q) = 0. \tag{2}$$

On the other hand, the relation (1) is a solution of the differential equation (2).

In books on the Calculus, it is shown that the equation, which results from eliminating a and b from (1) and

$$\frac{\partial \phi}{\partial a} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial b} = 0, \tag{3}$$

contains the equation of the envelope of the double infinity of surfaces defined by (1), whenever there is an envelope. Now, a characteristic property of the envelope of a family of surfaces is that its normal at each point coincides with the normal of the surface of the family through that point. It follows that the equation of the envelope of the family (1) is also a solution of (2). This solution is known as the *singular solution* of the differential equation. It may also be obtained by eliminating p and q from the differential equation (2) and

$$\frac{\partial f}{\partial p} = 0 \quad \text{and} \quad \frac{\partial f}{\partial q} = 0.* \tag{4}$$

---

* For example, see Goursat-Bourlet, *Équations aux dérivées partielles du premier ordre*, p. 24.

Also, as in the case of singular solutions of an ordinary differential equation of the first order, extraneous loci may be introduced by each of these processes.*

XIII·6. General, complete, particular solution. The principles discussed above can be employed to find still further solutions of the differential equation

$$f(x, y, z, p, q) = 0, \tag{1}$$

when a solution involving two arbitrary constants

$$\phi(x, y, z, a, b) = 0 \tag{2}$$

is known.

A relation between a and b in (2), e.g.

$$b = \psi(a), \tag{3}$$

selects a single infinity of surfaces from the double infinity defined by (2). If this single infinity of surfaces has an envelope, the equation of the latter must also be a solution of (1).† It is found by eliminating a and b from (2), (3), and

$$\frac{\partial \phi}{\partial a} + \psi' \frac{\partial \phi}{\partial b} = 0. \tag{4}$$

Corresponding to each choice of $\psi(a)$, determining a single infinity of surfaces having an envelope, we have a solution, which is known as a *particular solution* of the differential equation. Particular solutions are also obtained by assigning definite values to a and b in (2).

While the primitive involving two arbitrary constants is not the only solution of the partial differential equation of the first order to which it gives rise, all other solutions can, at least theoretically, be obtained from it. Hence, Lagrange called it a *complete solution* of the differential equation, and used the term *general solution* to apply to the aggregate of the particular solutions.

---

* See M. J. H. Hill, *Philosophical Transactions,* Vol. 183 (A), 1892, pp. 141–278.

† This follows also from the characteristic property of the envelope of a family of surfaces, mentioned in §XIII·5.

According to the existence theorem for the solution of a partial differential equation of the first order (§XII·3), the general solution, when expressed as a single relation, involves an arbitrary function. But only in the case of linear partial differential equations and in special forms of non-linear ones can such forms be found for the general solution (§§XIII·1, XIII·2). As a matter of fact, in most problems giving rise to the solution of partial differential equations, some particular solution satisfying given boundary conditions is sought. This is frequently obtained independently of the general solution. For, often the extreme generality of a general solution involving an arbitrary function makes it difficult to obtain the desired particular solution from the general solution, even if the latter can be found.

In §XII·1, we noted that, as far as partial differential equations of the first order are concerned, a primitive involving precisely two arbitrary constants plays an exceptional rôle. Correspondingly, we find that, in the case of a non-linear partial differential equation of the first order, a solution involving two arbitrary constants stands out prominently as a complete solution. But, it should be emphasized that there is no unique complete solution. *Any solution involving two arbitrary constants may be taken as a complete solution.*

We saw in (§XII·1) that a complete solution of $z^2(p^2 + q^2 + 1) = R^2$ is

$$(x - a)^2 + (y - b)^2 + z^2 = R^2,$$

which represents a family of spheres of radius R with centers in the plane $z = 0$. The envelope of these is the pair of planes $z^2 = R^2$, whose equation is obtained by eliminating a and b from

$$(x - a)^2 + (y - b)^2 + z^2 = R^2, x - a = 0, y - b = 0,$$

or by eliminating p and q from

$$z^2(p^2 + q^2 + 1) = R^2, \quad z^2p = 0, \quad z^2q = 0.$$

Thus, $z^2 = R^2$ is seen to be a singular solution.

The choice of a relation $b = \psi(a)$ results in selecting those spheres whose centers lie along the curve $y = \psi(x)$ in the plane $z = 0$. The envelope of these is obviously the tubular surface traced out by the motion of a sphere of radius R moving with its center on the curve $y = \psi(x)$ and along its entire length. In particular, if $\psi(x)$ is a linear function of x, the envelope is a cylinder. As an example, suppose $b = ha + k$. To find the corresponding solution, we must eliminate a from

$$(x - a)^2 + (y - ha - k)^2 + z^2 = R^2, \; (x - a) + h(y - ha - k) = 0.$$

From the second of these, we have

$$a = \frac{x + hy - hk}{1 + h^2}.$$

Putting this in the other equation, we have

$$\frac{h^2(hx - y + k)^2 + (hx - y + k)^2}{(1 + h^2)^2} = R^2 - z^2,$$

or

$$(hx - y + k)^2 = (1 + h^2)(R^2 - z^2),$$

which is obviously the equation of a circular cylinder whose axis is the line

$$y = hx + k, \quad z = 0.$$

Since this solution contains two arbitrary constants, it is a complete solution. (See Ex. 4, §XIII·7.)

As an exercise, the student should start with

$$(hx - y + k)^2 = (1 + h^2)(R^2 - z^2)$$

as a primitive and show that, when h and k are the arbitrary constants, the resulting partial differential equation is also $z^2(p^2 + q^2 + 1) = R^2$.

XIII·7. Method of Lagrange and Charpit. The following method for finding a complete solution is due to Lagrange. According to Lacroix, *Calcul différentiel et intégral*, vol. II, 2nd edition, p. 548, it was perfected by Charpit. The memoir of the latter was presented to the French Academy of Sciences in June, 1784, but was never published.

Given the differential equation

$$f(x, y, z, p, q) = 0, \tag{1}$$

the method consists in finding a second relation

$$\phi(x, y, z, p, q, a) = 0, \tag{2}$$

which involves either p or q or both of them in such a way that, when (1) and (2) are solved for p and q* and their values are substituted in the total differential equation

$$dz = p\,dx + q\,dy, \tag{3}$$

the latter is an integrable total differential equation (§X·1). In addition, ϕ must involve an arbitrary constant. The solution of (3) is a complete solution of (1), since it determines z as such a function of x and y that the values of p and q, obtained from it, together with z satisfy (1); besides, it involves two arbitrary constants. To obtain ϕ, we may proceed as follows:

Differentiating (1) and (2) with respect to x, we have

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\left(\frac{\partial p}{\partial x}\right)_{x,z} + \frac{\partial f}{\partial q}\left(\frac{\partial q}{\partial x}\right)_{x,z} = 0,\dagger \tag{4}$$

$$\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial p}\left(\frac{\partial p}{\partial x}\right)_{x,z} + \frac{\partial \phi}{\partial q}\left(\frac{\partial q}{\partial x}\right)_{x,z} = 0. \tag{5}$$

Similarly, differentiating with respect to y, we have

$$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\left(\frac{\partial p}{\partial y}\right)_{y,z} + \frac{\partial f}{\partial q}\left(\frac{\partial q}{\partial y}\right)_{y,z} = 0, \tag{6}$$

$$\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial p}\left(\frac{\partial p}{\partial y}\right)_{y,z} + \frac{\partial \phi}{\partial q}\left(\frac{\partial q}{\partial y}\right)_{y,z} = 0. \tag{7}$$

The condition that (3) shall be integrable is, in this notation,

$$\left(\frac{\partial p}{\partial y}\right)_{y,z} - \left(\frac{\partial q}{\partial x}\right)_{x,z} = 0. \tag{8}$$

From equations (4), (5), (6), (7), (8) we can eliminate the four quantities $(\partial p/\partial x)_{x,z}$, $(\partial p/\partial y)_{y,z}$, $(\partial q/\partial x)_{x,z}$, $(\partial q/\partial y)_{y,z}$. We elimi-

---

* This requires that no relation exist between f and ϕ, when considered as functions of p and q. The condition for the independence of f and ϕ is

$$\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial \phi}{\partial p} \neq 0.$$

See Note I in the Appendix.

† Here $\left(\frac{\partial u}{\partial x}\right)_{x,z}$ stands for $\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}$, and $\left(\frac{\partial u}{\partial y}\right)_{y,z}$ for $\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}$. For this notation, see the author's *Calculus*, p. 346.

nate $(\partial p/\partial x)_{x,z}$ by multiplying (4) and (5) by $\partial\phi/\partial p$ and $\partial f/\partial p$ respectively and taking their difference. The result is

$$(9)\quad \left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial\phi}{\partial p}-\left(\frac{\partial\phi}{\partial x}+p\frac{\partial\phi}{\partial z}\right)\frac{\partial f}{\partial p}+\left(\frac{\partial f}{\partial q}\frac{\partial\phi}{\partial p}-\frac{\partial\phi}{\partial q}\frac{\partial f}{\partial p}\right)\left(\frac{\partial q}{\partial x}\right)_{x,z}=0.$$

Multiplying (6) and (7) by $\partial\phi/\partial q$ and $\partial f/\partial q$ respectively and taking their difference, we eliminate $(\partial q/\partial y)_{y,z}$ and have

$$(10)\quad \left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial\phi}{\partial q}-\left(\frac{\partial\phi}{\partial y}+q\frac{\partial\phi}{\partial z}\right)\frac{\partial f}{\partial q}+\left(\frac{\partial f}{\partial p}\frac{\partial\phi}{\partial q}-\frac{\partial\phi}{\partial p}\frac{\partial f}{\partial q}\right)\left(\frac{\partial p}{\partial y}\right)_{y,z}=0.$$

Making use of (8), we have, on taking the sum of (9) and (10) and rearranging the terms,

$$(11)\quad \left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial\phi}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial\phi}{\partial q}-\frac{\partial f}{\partial p}\frac{\partial\phi}{\partial x}-\frac{\partial f}{\partial q}\frac{\partial\phi}{\partial y}$$
$$-\left(p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}\right)\frac{\partial\phi}{\partial z}=0.$$

This is a linear partial differential equation. Hence, to solve for ϕ, we consider the system of ordinary equations

$$(12)\quad \frac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\frac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}}=\frac{dx}{-\dfrac{\partial f}{\partial p}}=\frac{dy}{-\dfrac{\partial f}{\partial q}}=\frac{dz}{-\left(p\dfrac{\partial f}{\partial p}+q\dfrac{\partial f}{\partial q}\right)}.$$

We desire to find, not the most general form of ϕ, but some form of it that contains p or q and an arbitrary constant. In actual practice, a simple form is desirable.

REMARK. It is not difficult to remember (12). If, as before, we put

$$\left(\frac{\partial u}{\partial x}\right)_{x,z}=\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\quad\text{and}\quad\left(\frac{\partial u}{\partial y}\right)_{y,z}=\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z},$$

we have the skew symmetrical form

$$(12')\qquad \frac{dp}{\left(\dfrac{\partial f}{\partial x}\right)_{x,z}}=\frac{dx}{-\dfrac{\partial f}{\partial p}}=\frac{dq}{\left(\dfrac{\partial f}{\partial y}\right)_{y,z}}=\frac{dy}{-\dfrac{\partial f}{\partial q}};$$

the equality of each of these terms to $\dfrac{dz}{-\left(p\dfrac{\partial f}{\partial p}+q\dfrac{\partial f}{\partial q}\right)}$ results from $\dfrac{dx}{-\dfrac{\partial f}{\partial p}}=\dfrac{dy}{-\dfrac{\partial f}{\partial q}}$, as may be seen by composition, after multiplying numerator and denominator of the first by p and of the second by q and taking account of (3).

Exercise **1.** $z - pq = 0$.

To obtain ϕ, we must find a solution of

$$\frac{dp}{p}=\frac{dq}{q}=\frac{dx}{q}=\frac{dy}{p}=\cdots.$$

Using the first two members, we have

$$\phi \equiv p - aq = 0.$$

Solving this and the original equation for p and q, we have

$$p = a\sqrt{\frac{z}{a}},\ q = \sqrt{\frac{z}{a}}.$$

Then (3) becomes

$$dz = a\sqrt{\frac{z}{a}}\,dx + \sqrt{\frac{z}{a}}\,dy,$$

or

$$\frac{\sqrt{a}\,dz}{\sqrt{z}} = a\,dx + dy.$$

Integrating, we have

$$2\sqrt{az} = ax + y + b,$$

or

$$4az = (ax + y + b)^2, \tag{13}$$

which is a complete solution.

If we had used the second and third members, we would have had

$$\phi \equiv q - x - a = 0;$$

whence

$$q = x + a \quad \text{and} \quad p = \frac{z}{x+a}.$$

Then (3) becomes

$$dz = \frac{z}{x+a}\,dx + (x+a)dy,$$

or

$$dy = \frac{(x+a)dz - zdx}{(x+a)^2}.$$

Integrating, we have

$$y + b = \frac{z}{x+a},$$

or

$$z = (x+a)(y+b), \tag{14}$$

which is also a complete solution.

The general and singular solutions can be obtained from either (13) or (14). Thus, using the latter, we find the general solution as the aggregate of the particular solutions obtained by eliminating a from

$$z = (x+a)[y+\psi(a)]$$

and

$$y + \psi(a) + \psi'(a)(x+a) = 0,$$

where $\psi(a)$ is an arbitrarily selected function of a.

In particular, show that, if we put

$$\psi(a) = -ha + k,$$

where h and k are any constants, the corresponding solution is

$$4hz = (hx + y + k)^2.$$

This is (13).

The singular solution, obtained by eliminating a and b from

$$z = (x+a)(y+b), \quad y + b = 0, \quad x + a = 0,$$

is $z = 0$.

Let the student show that this can also be obtained from the other form of solution (13).

2. $p = (z + yq)^2$.

3. $\sqrt{p} + \sqrt{q} = 2x$.

4. $z^2(p^2 + q^2 + 1) = R^2$.

XIII·8. Transformation of variables. Attention should be called to the effect of each of several simple transformations of variables.

(*a*) Putting $X = \log x$, whence $\dfrac{dX}{dx} = \dfrac{1}{x}$, we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{dX}{dx} = \frac{1}{x}\frac{\partial z}{\partial X}.$$

$$\therefore xp = \frac{\partial z}{\partial X}; \text{ this does not involve } X.$$

(*b*) Putting $Y = \log y$, whence $\dfrac{dY}{dy} = \dfrac{1}{y}$, we have

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{dY}{dy} = \frac{1}{y}\frac{\partial z}{\partial Y}.$$

$$\therefore yq = \frac{\partial z}{\partial Y}; \text{ this does not involve } Y.$$

(*c*) Putting $Z = \log z$, whence $z = e^Z$, we have

$$p = \frac{\partial z}{\partial x} = e^Z\frac{\partial Z}{\partial x} = z\frac{\partial Z}{\partial x} \qquad \therefore \frac{p}{z} = \frac{\partial Z}{\partial x},$$

$$q = \frac{\partial z}{\partial y} = e^Z\frac{\partial Z}{\partial y} = z\frac{\partial Z}{\partial y} \qquad \therefore \frac{q}{z} = \frac{\partial Z}{\partial y}.$$

These do not involve Z.

Moreover, the simultaneous application of any two of these transformations, or of all three of them, has the joint effect of each of the transformations. Thus, the effect of

(*a*) and (*b*) is to replace xp by $\dfrac{\partial z}{\partial X}$ and yq by $\dfrac{\partial z}{\partial Y}$,

(*b*) and (*c*) is to replace $\dfrac{p}{z}$ by $\dfrac{\partial Z}{\partial x}$ and $\dfrac{yq}{z}$ by $\dfrac{\partial Z}{\partial Y}$,

(*a*), (*b*), (*c*) is to replace $\dfrac{xp}{z}$ by $\dfrac{\partial Z}{\partial X}$ and $\dfrac{yq}{z}$ by $\dfrac{\partial Z}{\partial Y}$,

and so on.

Use of these facts, at times, enables us to transform a given differential equation into one which does not involve one or more of the variables, thus simplifying the process of carrying out the Lagrange-Charpit method. As examples of such equations to which the Lagrange-Charpit method can be applied readily, the following types may be mentioned:

1° $f(p, q) = 0$. *All the variables are absent.*

To determine ϕ, the Lagrange-Charpit method leads to

$$\frac{dp}{0} = \cdots.$$

From this we have $p = a$.

Solving $f(a, q) = 0$ for q, we find $q =$ const. Calling this b, we have to solve finally

$$dz = a dx + b dy, \quad \text{where} \quad f(a, b) = 0.$$

Hence, a complete solution is

$$z = ax + by + c, \quad \text{where} \quad f(a, b) = 0.$$

Because of the ease with which a differential equation of this type can be solved, it is desirable to transform an equation to this form when practicable. From what we saw above concerning the effects of the transformations (a), (b), (c), it follows that equations of the form $f(xp, q) = 0$, $f(xp, yq) = 0$, $f(p/z, q/z) = 0$, $f(p/z, yq/z) = 0$, and the like, can be reduced readily to type 1° by one or several of these transformations.

***Exercise* 1.** $x^2p^2 + q^2 = z^2$.

If this is written in the form

$$\left(\frac{xp}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = 1,$$

it is evident that the transformations $X = \log x$, $Z = \log z$ reduce the equation to

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = 1,$$

which is free of all the variables, and is, thus, of type 1°. A complete solution is

$$Z = aX + by + c, \quad \text{where} \quad a^2 + b^2 = 1,$$

or

$$\log z = a \log x + by + c, \quad \text{where} \quad a^2 + b^2 = 1.$$

This may also be written

$$z = cx^a e^{by}, \quad \text{where} \quad a^2 + b^2 = 1.$$

Here we have replaced the arbitrary constant e^c by c, instead of using another letter.

Because of the given relation between a and b, these may be replaced by the quantities $\sin \alpha$ and $\cos \alpha$ respectively, where α is an arbitrary constant. The solution then takes the form

$$z = cx^{\sin \alpha} e^{y \cos \alpha}.$$

2. $pq = z^2$. **3.** $x^2p^2 - q^2 = 1$.

4. $ypq = p^2 + z^2$.

2° $f(z, p, q) = 0$. *Both of the independent variables are absent.* To determine ϕ, the Lagrange-Charpit method leads to

$$\frac{dp}{p\frac{\partial f}{\partial z}} = \frac{dq}{q\frac{\partial f}{\partial z}} = \cdot \cdot \cdot.$$

From this we have $p = aq$. Substituting this value of p in the differential equation, we have

$$f(z, aq, q) = 0.$$

Solving for q, we find it to be a function of z and a, e.g. $q = \psi(z, a)$; and, therefore, $p = a\psi(z, a)$. Finally, we must solve the total differential equation

$$dz = a\psi(z, a)dx + \psi(z, a)dy.$$

Divided by $\psi(z, a)$, this becomes exact and can be solved by a quadrature.

This suggests the advisability of using transformations (a) and (b) in the case of equations of the form $f(z, xp, q) = 0$, $f(z, xp, yq) = 0$, $f(z, p, yq) = 0$.

5. $xp(1 + q) = qz$. **6.** $xypq = z$.

3° $f_1(x, p) = f_2(y, q)$. *The dependent variable is absent and there is a quasi separation of variables.*

To determine ϕ, we have

$$\frac{dp}{\frac{\partial f_1}{\partial x}} = \cdots = \frac{dx}{-\frac{\partial f_1}{\partial p}} = \cdots.$$

This leads to

$$\frac{\partial f_1}{\partial x}\,dx + \frac{\partial f_1}{\partial p}\,dp = 0, \quad \text{or} \quad df_1 = 0.$$

Hence, $f_1(x, p) = a$; and, therefore, $f_2(y, q) = a$.

Solving these respectively for p and q, we find

$$p = \psi_1(x, a) \quad \text{and} \quad q = \psi_2(y, a).$$

The resulting total differential equation

$$dz = \psi_1(x, a)dx + \psi_2(y, a)dy$$

is exact.

Evidently, if we have an equation of the form

$$f_1\left(x, \frac{p}{z}\right) = f_2\left(y, \frac{q}{z}\right),$$

it is usually desirable to employ the transformation (c) before proceeding to use the Lagrange-Charpit method.

7. $zq = 2yp^2$.

8. $xzp - yz^2 = q^2$.

Remark. While the method suggested for finding a complete solution of an equation of type 3° seems the obvious one, in some cases it may be preferable to find ϕ by using some other differential equation of the system of equations (12) of §XIII·7. Thus, consider

9. $p - yq = q^2$.

It is possible to solve $q^2 + yq = a$ for q and to proceed to a complete solution. However, if use is made of the fact that the system of differential equations (12) of §XIII·7 also includes

$$\cdots = \frac{dq}{-q} = \frac{dx}{-1} = \cdots,$$

a more desirable form of complete solution results.

XIII·9. Extended Clairaut equation. Recalling the form of the solution of a Clairaut equation (§IV·4), one can find, by inspection, a complete solution of the *extended Clairaut equation*

$$z = xp + yq + f(p, q)$$

in the form

$$z = ax + by + f(a, b).$$

It is easy to verify that this is a solution of the differential equation.

This solution is the equation of ∞^2 planes. Usually the equation of the envelope of these planes, which is the singular solution of the differential equation, is of interest.

Exercise **1.** $z = xp + yq + \sqrt{p^2 + q^2 + 1}$.

2. $z = xp + yq + p^2 + q^2$.

3. $z = xp + qy + \dfrac{1}{p^2 + q^2}$.

4. $z = xp + yq + p^2 + pq + q^2$.

XIII·10. Summary. Partial differential equations of the first order may be grouped into two general classes: those which are linear in the derivatives of the dependent variable, and those which are not.

1° For the solution of linear partial differential equations of the first order, the method of Lagrange applies, giving the general solution (§XIII·1).

Special solutions may occur when the coefficients of a linear partial differential equation are not rational (§XIII·2).

2° For the solution of non-linear equations of the first order, the general method of Lagrange and Charpit applies (§XIII·7), giving a complete solution. From this the other solutions can be obtained (§§XIII·5 and XIII·6).

A transformation of variables may aid in solving a differential equation. The choice of new variables is frequently suggested by the form of the differential equation. In §XIII·8 transformations are discussed, as a result of which one or more of the new variables are absent from the transformed equation. Such simplification of the differential equation is usually desirable before applying the Lagrange-Charpit method.

A complete solution for an extended Clairaut equation is found most readily by the method of §XIII·9. In this case, the singular solution is usually of importance.

Partial differential equations of the first order, in which only one of the partial derivatives appears, may be solved as ordinary differential equations (§XIII·2).

Exercise **1.** $x^2p + y^2q = z^2$. **2.** $x^2p^2 + y^2q^2 = z^2$.

3. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = xyz$. **4.** $z = px + qy + pq$.

5. $xypq = z^2$. **6.** $y^2zp + x^2zq = y^2x$.

7. $q = xp + x^2p^2$. **8.** $(p + q)(px + qy) = 1$.

9. $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.
[Let $x + y = u^2$, $x - y = v^2$.]

10. $(p^2 + q^2)x - pz = 0$.

11. $(x^2 + y^2)(p^2 + q^2) = 1$. [Let $x = \rho \cos \theta$, $y = \rho \sin \theta$.]

12. $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$.

13. $q^2 = z^2(p - q)$.

14. $(y - x)(qy - px) = (p - q)^2$. [Let $xy = u$, $x + y = v$.]

15. $z - xp - yq = n\sqrt{x^2 + y^2 + z^2}$.

16. $pq = px + qy$.

17. $$(y + z + u)\frac{\partial u}{\partial x} + (z + u + x)\frac{\partial u}{\partial y} + (u + x + y)\frac{\partial u}{\partial z} = x + y + z.$$

18. $yq = x^2p^2 + 1$. **19.** $p^2 + q^2 = (x + y)z^2$.

20. $z^2(p^2 + q^2) = x^2 + y^2$. [Let $xy = u$, $x^2 - y^2 = 2v$.]

21. $(x^2 - y^2)pq - xy(p^2 - q^2) = 1$. [Introduce polar coördinates.]

22. $yzp^2 = q$. **23.** $x^2 + (z - yq)^2 = 1$.

24. $p^{1/2} + q^{1/2} = x$. **25.** $z = px + qy + 3x^{1/3}y^{1/3}$.

26. Determine a family of surfaces such that the normal at each point makes a constant angle with the plane of xy.

27. Determine a family of surfaces such that the coördinates of the point where the normal meets the plane of xy are proportional to the corresponding coördinates of the point on the surface.

28. Determine a family of surfaces for which the product of the distances from two fixed points to a tangent plane of a surface is the same for all the tangent planes of that surface.

29. Determine a family of surfaces for which the product of the intercepts of a tangent plane of a surface on the coördinate axes is the same for all the tangent planes of that surface.

30. Determine a family of surfaces for which the sum of the intercepts of a tangent plane of a surface is the same for all the tangent planes of that surface.

NOTE. In regard to exercises 26–30, it should be recalled that

(*a*) $p/\sqrt{p^2+q^2+1}$, $q/\sqrt{p^2+q^2+1}$, $-1/\sqrt{p^2+q^2+1}$ are the direction cosines of the normal to the surface $z = f(x, y)$, or $F(x, y, z)$, at (x, y, z);

(*b*) $\dfrac{X-x}{p} = \dfrac{Y-y}{q} = \dfrac{Z-z}{-1}$ are the equations of the normal to the surface at (x, y, z), where X, Y, Z are the coördinates of a variable point on the normal;

(*c*) $(x+pz, y+qz)$ is the point where the normal meets the xy-plane;

(*d*) $p(X-x)+q(Y-y) = Z-z$ is the equation of the tangent plane to the surface at (x, y, z), where X, Y, Z are the coördinates of a variable point in the tangent plane;

(*e*) $\dfrac{p(X_0-x)+q(Y_0-y)-(Z_0-z)}{\sqrt{p^2+q^2+1}}$ is equal to the distance of the point (X_0, Y_0, Z_0) from the tangent plane;

(*f*) $(xp+yq-z)/p$, $(xp+yq-z)/q$, $-(xp+yq-z)$ are the lengths of the intercepts of the tangent plane on the axes of x, y, z respectively.

CHAPTER XIV

PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER THAN THE FIRST

XIV·1. Linear partial differential equation. We shall first consider partial differential equations which are *linear in the dependent variable and all of its derivatives.* The general type of such an equation is

$$(1)\quad P_{n,0}\frac{\partial^n z}{\partial x^n}+P_{n-1,1}\frac{\partial^n z}{\partial x^{n-1}\partial y}+P_{n-2,2}\frac{\partial^n z}{\partial x^{n-2}\partial y^2}+\cdots+P_{0,n}\frac{\partial^n z}{\partial y^n}$$
$$+P_{n-1,0}\frac{\partial^{n-1}z}{\partial x^{n-1}}+\cdots+P_{s,r}\frac{\partial^{r+s}z}{\partial x^s\partial y^r}+\cdots+P_{1,0}\frac{\partial z}{\partial x}$$
$$+P_{0,1}\frac{\partial z}{\partial y}+P_{0,0}z=f(x,y),$$

where the coefficients are functions of x and y. If we put $\partial/\partial x = D$ and $\partial/\partial y = \mathfrak{D}$, (1) may be written in the form

$$(P_{n,0}D^n+P_{n-1,1}D^{n-1}\mathfrak{D}+P_{n-2,2}D^{n-2}\mathfrak{D}^2+\cdots+P_{0,n}\mathfrak{D}^n+P_{n-1,0}D^{n-1}$$
$$+\cdots+P_{s,r}D^s\mathfrak{D}^r+\cdots+P_{1,0}D+P_{0,1}\mathfrak{D}+P_{0,0})z=f(x,y),$$

or, more briefly,

$$(1)\qquad F(D,\mathfrak{D})z=f(x,y),$$

where $F(D, \mathfrak{D})$ is a symbolic operator, which, looked upon algebraically, is a polynomial of degree n in D and $\mathfrak{D}$. There are many points of similarity between this equation and the linear ordinary differential equation of the nth order (§VI·1).

Obviously,

$$F(D,\mathfrak{D})(u+v)=F(D,\mathfrak{D})u+F(D,\mathfrak{D})v.$$

Hence, as in the case of linear ordinary differential equations, the problem of solving (1) can be divided into two parts, viz. that of finding the general integral of

$$(2)\qquad F(D,\mathfrak{D})z=0,$$

which we shall call the *complementary function* of (1), and that of finding any particular integral. The sum of these will give the general integral of (1).

XIV·2. Homogeneous linear equation with constant coefficients. Following a generally adopted convention, we shall use the term *homogeneous* to apply to a linear partial differential equation in which all the derivatives are of the same order. In this case the symbolic operator, $F(D, \mathfrak{D})$, is homogeneous in D and $\mathfrak{D}$. Suppose, besides, that the coefficients are constants and the right-hand member is zero. Our equation will be of the form

$$(k_0 D^n + k_1 D^{n-1}\mathfrak{D} + \cdots + k_{n-1} D\mathfrak{D}^{n-1} + k_n \mathfrak{D}^n)z = 0, \tag{1}$$

or

$$F(D, \mathfrak{D})z = 0.$$

If ϕ is a function of $y + mx$,

$$D^r\mathfrak{D}^s\phi(y + mx) = m^r\phi^{(r+s)}(y + mx),$$

where $\phi^{(r+s)}(y+mx)$ means $\dfrac{d^{r+s}\phi(y + mx)}{[d(y + mx)]^{r+s}}$. Consequently, the result of substituting $z = \phi(y + mx)$ in (1) is

$$\phi^{(n)}(y + mx)F(m, 1) = 0.$$

Hence, $z = \phi(y + mx)$ will be a solution, provided $F(m, 1) = 0$; i.e., provided

$$k_0 m^n + k_1 m^{n-1} + \cdots + k_{n-1} m + k_n = 0. \tag{2}$$

If the roots of (2), which we shall speak of as the *auxiliary equation*, are distinct, e.g. $m_1, m_2 \cdots, m_n$,

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + m_n x)$$

will be a solution. Since it contains n arbitrary functions, it is the general solution.*

---

* If $F(D, \mathfrak{D})$ contains $\mathfrak{D}$ as a factor, $F(m, 1)$ is only of degree $n - 1$. The lost root in this case is ∞, and the corresponding integral is $\phi(x)$. This is also obvious from the form of the differential equation in this case. For, to say that $\mathfrak{D}$ is a factor of $F(D, \mathfrak{D})$ means that every derivative of z is taken at least

Exercise **1.** $\dfrac{\partial^2 z}{\partial x^2} - a^2 \dfrac{\partial^2 z}{\partial y^2} = 0.$

The auxiliary equation in this case is

$$m^2 - a^2 = 0. \quad \therefore m = \pm a.$$

Hence the general solution is

$$z = \phi(y + ax) + \psi(y - ax).$$

2. $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 0.$ **3.** $\dfrac{\partial^3 z}{\partial x^3} - 7\dfrac{\partial^3 z}{\partial x^2 \partial y} + 10\dfrac{\partial^3 z}{\partial x \partial y^2} = 0.$

4. $\dfrac{\partial^3 z}{\partial x^2 \partial y} - 2\dfrac{\partial^3 z}{\partial x \partial y^2} - 3\dfrac{\partial^3 z}{\partial y^3} = 0.$

XIV·3. Roots of auxiliary equation repeated. If any of the roots of the auxiliary equation are repeated, the method of §XIV·2 fails to give us the general solution. In this case, we proceed by a method entirely analogous to that in §VI·7.

The symbolic operator $F(D, \mathfrak{D})$ may be written as the product of its factors

$$(D - m_1\mathfrak{D})(D - m_2\mathfrak{D}) \cdot \cdot \cdot (D - m_n\mathfrak{D}).$$

Moreover, it is readily seen, in a manner similar to that employed in §VI·6, that the order of these factors is immaterial. Suppose that m_1 is a repeated root. We wish to find a solution of

$$(D - m_1\mathfrak{D})(D - m_1\mathfrak{D})z = 0.$$

If we let $(D - m_1\mathfrak{D})z = v$, our equation becomes

$$(D - m_1\mathfrak{D})v = 0.$$

Hence, by the method of §XIV·2 or of §XIII·1,

$$v = \phi(y + m_1 x).$$

We now have to solve

$$(D - m_1\mathfrak{D})z = \phi(y + m_1 x).$$

---

once with respect to y. Hence, $F(D, \mathfrak{D})\phi(x) = 0$. Similarly, if $F(D, \mathfrak{D})$ contains $\mathfrak{D}^r$ as a factor, $\phi_1(x), y\phi_2(x), \cdot \cdot \cdot, y^{r-1}\phi_r(x)$ are readily seen to be integrals.

All cases could have been accounted for at one time by selecting ϕ as an arbitrary function of $mx + ny$, where, now, only the ratio m/n or n/m plays a rôle. But the method used in the text, including the first part of this footnote, seems simpler in practice.

This is linear, of the first order, and can be written in the form

$$p - m_1 q = \phi(y + m_1 x).$$

Hence the method of Lagrange (§XIII·1) applies. Employing it, we have

$$\frac{dx}{1} = -\frac{dy}{m_1} = \frac{dz}{\phi(y + m_1 x)}.$$

From the first two members we have

$$y + m_1 x = a.$$

Putting this in the last member, we have

$$dx = \frac{dz}{\phi(a)}.$$

$$\therefore x\phi(a) - z = b.$$

Hence the general solution is

$$x\phi(y + m_1 x) - z = \psi(y + m_1 x),$$

or

$$z = \psi(y + m_1 x) + x\phi(y + m_1 x).$$

In other words, *if m_1 is a double root of the auxiliary equation, not only is $\phi(y + m_1 x)$ an integral, but so also is $x\psi(y + m_1 x)$.*

In an entirely analogous manner it can be shown that, *if m_1 is an r-fold root,*

$$\phi_1(y + m_1 x),\ x\phi_2(y + m_1 x),\ x^2\phi_3(y + m_1 x), \cdots, x^{r-1}\phi_r(y + m_1 x)$$

are all integrals.

By a proof similar to that employed above, we can show that

$$\phi_1(y + m_1 x),\ y\phi_2(y + m_1 x),\ \cdots,\ y^{r-1}\phi_r(y + m_1 x)$$

is also a set of r independent integrals in this case.

***Exercise* 1.** $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

2. $\dfrac{\partial^3 z}{\partial x^3} + \dfrac{\partial^3 z}{\partial x^2 \partial y} - \dfrac{\partial^3 z}{\partial x \partial y^2} - \dfrac{\partial^3 z}{\partial y^3} = 0.$ **3.** $\dfrac{\partial^4 z}{\partial x^4} - 2\dfrac{\partial^4 z}{\partial x^2 \partial y^2} + \dfrac{\partial^4 z}{\partial y^4} = 0.$

XIV·4. Roots of auxiliary equation complex. If the coefficients in the differential equation are real, the complex roots of the auxiliary equation occur in pairs of conjugates. Hence, if $\alpha + i\beta$ is a root, $\alpha - i\beta$ will also be one. The corresponding terms in the complementary function will be

$$(1) \qquad \phi(y + \alpha x + i\beta x) + \psi(y + \alpha x - i\beta x).$$

If we let

$$\phi = \phi_1 + i\psi_1 \quad \text{and} \quad \psi = \phi_1 - i\psi_1,$$

where ϕ_1 and ψ_1 are any two arbitrarily chosen functions, the expression (1) becomes

$$\phi_1(y + \alpha x + i\beta x) + \phi_1(y + \alpha x - i\beta x) \\ + i[\psi_1(y + \alpha x + i\beta x) - \psi_1(y + \alpha x - i\beta x)].$$

For ϕ_1 and ψ_1 any real functions, this is real.

Exercise. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 0.$

The auxiliary equation is

$$m^2 - 2m + 2 = 0. \quad \therefore m = 1 \pm i.$$

The general solution is

$$z = \phi(y + x + ix) + \psi(y + x - ix).$$

It will assume a real form

$$z = \phi_1(y + x + ix) + \phi_1(y + x - ix) \\ + i[\psi_1(y + x + ix) - \psi_1(y + x - ix)],$$

for ϕ_1 and ψ_1 any real functions.

For example, if we choose $\phi_1(u)$ to be $\cos u$ and $\psi_1(u)$ to be e^u, we have

$$\begin{aligned} \cos(y + x + ix) &= \cos(x + y)\cos ix - \sin(x + y)\sin ix \\ &= \cos(x + y)\cosh x - i\sin(x + y)\sinh x, \\ \cos(y + x - ix) &= \cos(x + y)\cos ix + \sin(x + y)\sin ix \\ &= \cos(x + y)\cosh x + i\sin(x + y)\sinh x. \end{aligned}$$

$$e^{y+x+ix} - e^{y+x-ix} = e^{y+x}(e^{ix} - e^{-ix}) = 2ie^{y+x}\sin x.$$

$$\therefore z = 2\cos(x + y)\cosh x - 2e^{x+y}\sin x.$$

XIV·5. Particular integral. General methods for finding the particular integral which must be added to the complementary function to obtain the general integral, in case the right-hand member of the equation is different from zero, may be deduced along lines entirely analogous to those for linear ordinary differential equations with constant coefficients (§§VI·7, VI·8). In a large number of cases, these can be found more simply by trial, employing essentially the same method as that of undetermined coefficients (§VI·10). The following exercises will illustrate:

***Exercise* 1.** $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 2\dfrac{\partial^2 z}{\partial y^2} = \sin\,(x + 2y) - 2\,\sin\,(x + y) + x + xy.$

The complementary function is $\phi(y + x) + \psi(y - 2x)$.

To obtain the part of the particular integral corresponding to $\sin\,(x + 2y)$, we try $z = a \sin\,(x + 2y)$, since all the derivatives are of the second order. Then

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 y}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 5a \sin\,(x + 2y).$$

If $a = \frac{1}{5}$, this becomes $\sin\,(x + 2y)$. Hence the required particular integral is $\frac{1}{5} \sin\,(x + 2y)$.

Since $\sin\,(x + y)$ is part of the complementary function, it is useless to try $z = b \sin\,(x + y)$ in order to obtain $-2 \sin\,(x + y)$, the next term in the right-hand member. Trying $z = bx \sin\,(x + y)$, we get $3b \cos\,(x + y)$. Hence, we must try $z = bx \cos\,(x + y)$. Doing this, we have as a result of substituting in the equation $-3b \sin\,(x + y)$. This will be $-2 \sin\,(x + y)$ if $b = \frac{2}{3}$. Hence, $\frac{2}{3}x \cos\,(x + y)$ is the required particular integral for this term. [It will be left as an exercise to show that we might also have used $z = by \cos\,(x + y)$.]

For the term x we try $z = cx^3$. Substituting this, we have $6cx$. This equals x if $6c = 1$; hence the corresponding particular integral is $x^3/6$.

For xy we try $z = fx^3y$. Using this, we have $6fxy + 3fx^2$. So we try $z = fx^3y + gx^4$. Using this, we have $6fxy + (3f + 12g)x^2$. This equals xy if $f = \frac{1}{6}$, $g = -\frac{1}{24}$. Hence the required particular integral for this term is $\frac{1}{6}x^3y - \frac{1}{24}x^4$. And the general solution is

$$z = \phi(y + x) + \psi(y - 2x) + \tfrac{1}{5}\sin(x + 2y) + \tfrac{2}{3}x\cos(x + y) + \tfrac{1}{6}x^3 + \tfrac{1}{6}x^3y - \tfrac{1}{24}x^4.$$

Exercise **2.** $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = e^{x+2y} + e^{x+y}.$

The complementary function is $\phi(y + x) + \psi(y + 2x)$.

For e^{x+2y} we try $z = ae^{x+2y}$. Using this, we have $3ae^{x+2y}$. Hence, $\frac{1}{3}e^{x+2y}$ is the particular integral desired.

Since e^{x+y} is a part of the complementary function, we shall try $z = bxe^{x+y}$. Using this, we have $-be^{x+y}$. Hence, $-xe^{x+y}$ is the particular integral desired. [Of course, we might have used $z = bye^{x+y}$ instead.] And the general solution is

$$z = \phi(y + x) + \psi(y + 2x) + \tfrac{1}{3}e^{x+2y} - xe^{x+y}.$$

Exercise **3.** $\dfrac{\partial^3 z}{\partial x^2 \partial y} - 2\dfrac{\partial^3 z}{\partial x \partial y^2} + \dfrac{\partial^3 z}{\partial y^3} = \dfrac{1}{x^2}.$

The auxiliary equation is $m^2 - 2m + 1 = 0$. $\therefore m = 1, 1$.

The complementary function is $\phi(x) + \psi(y + x) + x\chi(y + x)$.

Here, there is no term in $\partial^3 z/\partial x^3$. Hence, in order to get $1/x^2$ we may take y times a function of x whose second derivative is a constant times $1/x^2$; that is, we shall try $z = ay \log x$. Doing this, we obtain $1/x^2$ when $a = -1$. Hence the general solution is

$$z = \phi(x) + \psi(y + x) + x\chi(y + x) - y \log x.$$

4. $2\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} - 3\dfrac{\partial^2 z}{\partial y^2} = \sin(x + y) + 3x^2y.$

5. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} - 2\dfrac{\partial^2 z}{\partial y^2} = x - y.$

XIV·6. Non-homogeneous linear equation with constant coefficients. If $F(D, \mathfrak{D})$ of §XIV·1 is not homogeneous in D and $\mathfrak{D}$, but the coefficients are constants, it is only under certain circumstances that we can obtain solutions involving arbitrary functions, although we can usually find solutions with an indefinite number of arbitrary constants.

Since $D^r e^{ax+by} = a^r e^{ax+by}$, $\mathfrak{D}^s e^{ax+by} = b^s e^{ax+by}$, the result of substituting $z = ce^{ax+by}$ in the left-hand member of

$$F(D, \mathfrak{D})z = 0 \tag{1}$$

is $ce^{ax+by} F(a, b)$. If a and b satisfy the relation

$$F(a, b) = 0, \tag{2}$$

which we shall call the *auxiliary equation*, $z = ce^{ax+by}$ will be a solution of (1) where c is any constant. Corresponding to any value of b there will be a definite number of values of a satisfying (2). Hence we can find as many particular solutions as we need by giving various values to b. Now, the sum of any number of integrals of (1) is also an integral. Hence

$$z = \sum ce^{ax+by} \tag{3}$$

is a solution where the c's and b's are arbitrary constants, indefinite in number, and each a is so chosen that with the corresponding b it satisfies (2).

If, corresponding to any value of b, we have the k values of a satisfying (2) in the form $f_1(b), f_2(b), \cdots, f_k(b)$, we can write (3) in the form

$$z = \sum ce^{f_1(b)x+by} + \sum ce^{f_2(b)x+by} + \cdots + \sum ce^{f_k(b)x+by}, \tag{4}$$

where the c's and the b's are arbitrary.

In general, $F(D, \mathfrak{D})$ has no rational factors. If there is a linear factor $D - \lambda\mathfrak{D} - \mu$, the equation (2) will contain the factor $a - \lambda b - \mu$, whence $a = \lambda b + \mu$ Hence one of the f's, e.g. $f_1(b)$, becomes $\lambda b + \mu$, and the corresponding set of terms in (4) may be written

$$z = \sum ce^{b(\lambda x+y)+\mu x} = e^{\mu x}\sum ce^{b(\lambda x+y)}. \tag{5}$$

Since the c's and the b's are arbitrary, $\sum ce^{b(\lambda x+y)}$ is an arbitrary function of $\lambda x + y$, e.g. $\phi(\lambda x + y)$. Then, (5) may be written in the form

$$z = e^{\mu x}\phi(\lambda x + y). \tag{6}$$

Hence we see that, corresponding to every distinct linear factor of (2), we have a solution of the form (6).*

---

* In the case of the homogeneous equation (§XIV·2) all the factors are linear. Besides, in that case, $\mu = 0$, and the result there obtained coincides with what we have found here.

If there is a linear factor of $F(D, \mathfrak{D})$ which is free of D, the corresponding factor in (2) will be free of a; let it be $b - \mu$. The corresponding set of terms in (4) may be written

$$z = \sum ce^{ax+\mu y} = e^{\mu y}\sum ce^{ax}. \tag{5'}$$

Since the c's and a's are arbitrary, $\sum ce^{ax}$ is an arbitrary function of x, and (5′) may be written in the form

$$z = e^{\mu y}\phi(x). \tag{6'}$$

If the right-hand member of the differential equation is not zero, a particular integral may, at times, be found by trial as in §XIV·5.

Exercise 1. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial y} - z = \sin(x + 2y).$

The auxiliary equation (2) is

$$a^2 + ab + b - 1 = 0,$$

or

$$(a + 1)(a + b - 1) = 0.$$

Using $a + 1 = 0$, we have $\lambda = 0$, $\mu = -1$, and from (6) we see that $z = e^{-x}\phi(y)$ is a solution of the differential equation when the right-hand member is replaced by zero.

Using $a + b - 1 = 0$, we have $\lambda = -1$, $\mu = 1$. Hence a second solution is $z = e^x\psi(y - x)$. The complementary function is, therefore,

$$e^{-x}\phi(y) + e^x\psi(y - x).$$

For a particular integral try $z = \alpha\sin(x + 2y) + \beta\cos(x + 2y)$. Substituting this in the left-hand member, we have

$$(-4\alpha - 2\beta)\sin(x + 2y) + (-4\beta + 2\alpha)\cos(x + 2y).$$

This will equal $\sin(x + 2y)$ if $\alpha = -\frac{1}{5}$, $\beta = -\frac{1}{10}$. Hence a particular integral is

$$-\tfrac{1}{5}\sin(x + 2y) - \tfrac{1}{10}\cos(x + 2y),$$

and the general solution is

$$z = e^{-x}\phi(y) + e^x\psi(y - x) - \tfrac{1}{10}[2\sin(x + 2y) + \cos(x + 2y)].$$

Exercise **2.** $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} + 2\dfrac{\partial z}{\partial x} + z = e^{-x}.$

The auxiliary equation (2) is

$$a^2 - b^2 + 2a + 1 = 0,$$

or

$$(a + b + 1)(a - b + 1) = 0.$$

Hence the complementary function is

$$e^{-x}[\phi(y - x) + \psi(y + x)].$$

Since e^{-x} is part of the complementary function, we would naturally try $z = \alpha x e^{-x}$. But, this is also part of the complementary function, as may be seen by putting $\phi = -\dfrac{y - x}{2}$, $\psi = \dfrac{y + x}{2}$. We must then try $z = \alpha x^2 e^{-x}$. Substituting this in the left-hand member, we have $2\alpha e^{-x}$. Hence, $\alpha = \frac{1}{2}$, and the particular integral desired is $\frac{1}{2}x^2 e^{-x}$. The general solution is

$$z = e^{-x}[\phi(y - x) + \psi(y + x) + \tfrac{1}{2}x^2].$$

3. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} - 2\dfrac{\partial^2 z}{\partial y^2} + 2\dfrac{\partial z}{\partial x} + 2\dfrac{\partial z}{\partial y} = e^{2x+3y} + \sin(2x + y).$

4. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial y} - z = \cos(x + 2y) + e^y.$

XIV·7. Equation reducible to linear equations with constant coefficients. If the coefficient of $D^r\mathfrak{D}^s$ in $F(D, \mathfrak{D})$ of §XIV·1 is a constant times $x^r y^s$, the equation can be reduced to one with constant coefficients by the transformation $\log x = X$, $\log y = Y$. (Compare method employed in §VI·11.) Thus, e.g.

$$\begin{array}{ll}
Dz = \dfrac{\partial z}{\partial x} = \dfrac{1}{x}\dfrac{\partial z}{\partial X} & \therefore\ xDz = \dfrac{\partial z}{\partial X}. \\[2ex]
D^2 z = \dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{x^2}\dfrac{\partial^2 z}{\partial X^2} - \dfrac{1}{x^2}\dfrac{\partial z}{\partial X} & \therefore\ x^2 D^2 z = \dfrac{\partial^2 z}{\partial X^2} - \dfrac{\partial z}{\partial X}. \\[2ex]
\mathfrak{D}z = \dfrac{\partial z}{\partial y} = \dfrac{1}{y}\dfrac{\partial z}{\partial Y} & \therefore\ y\mathfrak{D}z = \dfrac{\partial z}{\partial Y}. \\[2ex]
\mathfrak{D}^2 z = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{y^2}\dfrac{\partial^2 z}{\partial Y^2} - \dfrac{1}{y^2}\dfrac{\partial z}{\partial Y} & \therefore\ y^2\mathfrak{D}^2 z = \dfrac{\partial^2 z}{\partial Y^2} - \dfrac{\partial z}{\partial Y}. \\[2ex]
D\mathfrak{D}z = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{xy}\dfrac{\partial^2 z}{\partial X \partial Y} & \therefore\ xyD\mathfrak{D}z = \dfrac{\partial^2 z}{\partial X \partial Y}. \\[2ex]
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}$$

Exercise **1.** $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x\partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = x^2 + y^2.$

If we let $\log x = X$, $\log y = Y$, the equation becomes

$$\frac{\partial^2 z}{\partial X^2} + 2\frac{\partial^2 z}{\partial X\partial Y} + \frac{\partial^2 z}{\partial Y^2} - \frac{\partial z}{\partial X} - \frac{\partial z}{\partial Y} = e^{2X} + e^{2Y},$$

which has constant coefficients. The auxiliary equation (2), §XIV·6, is

$$a^2 + 2ab + b^2 - a - b = 0, \quad \text{or} \quad (a + b)(a + b - 1) = 0.$$

Hence the complementary function is $\phi(Y - X) + e^X\psi(Y - X)$.

For the particular integral try $z = \alpha e^{2X} + \beta e^{2Y}$. Doing this, we find that $\alpha = \beta = \frac{1}{2}$. Hence the general solution is

$$z = \phi(Y - X) + e^X\psi(Y - X) + \tfrac{1}{2}(e^{2X} + e^{2Y}).$$

Passing back to x and y, and remembering that $Y - X = \log y/x$, we obtain the general solution in the form

$$z = \phi_1\left(\frac{y}{x}\right) + x\psi_1\left(\frac{y}{x}\right) + \tfrac{1}{2}(x^2 + y^2).$$

2. $x^2\dfrac{\partial^2 z}{\partial x^2} - y^2\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 0.$

3. $x^2\dfrac{\partial^2 z}{\partial x^2} - 4xy\dfrac{\partial^2 z}{\partial x\partial y} + 4y^2\dfrac{\partial^2 z}{\partial y^2} + 6y\dfrac{\partial z}{\partial y} = x^3y^4.$

Other equations may be reducible to linear equations with constant coefficients. (But the transformation is not always as obvious as in the case cited above.) Thus, let the student apply the transformation $X = \frac{1}{2}x^2$, $Y = \frac{1}{2}y^2$ to the following exercise.

4. $\dfrac{1}{x^2}\dfrac{\partial^2 z}{\partial x^2} - \dfrac{1}{x^3}\dfrac{\partial z}{\partial x} = \dfrac{1}{y^2}\dfrac{\partial^2 z}{\partial y^2} - \dfrac{1}{y^3}\dfrac{\partial z}{\partial y}.$

XIV·8. Particular solution. In the case of some important physical problems, where a particular solution is desired satisfying certain initial or boundary conditions, the general solution, involving arbitrary functions, is not as serviceable as the exponential form of solution, involving arbitrary constants, mentioned in §XIV·6. The following exercises will illustrate:

***Exercise* 1.** $\dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0.$*

Required to find a solution, which does not grow indefinitely, and which vanishes when $t = 0$, also when $x = 0$.

Using the method of §XIV·6, we find that $u = ce^{\alpha x + \beta t}$ will be a solution if

$$\beta^2 - a^2\alpha^2 = 0, \quad \text{or} \quad \beta = \pm a\alpha.$$

Hence we may take

$$u = e^{\alpha x}(c_1 e^{a\alpha t} + c_2 e^{-a\alpha t}),$$

where c_1, c_2, α are arbitrary constants. Since u must be finite for $t = \infty$, α must be imaginary. Let $\alpha = i\mu$. Then

$$u = e^{i\mu x}(c_1 e^{ia\mu t} + c_2 e^{-ia\mu t}).$$

As was done in §VI·4, this may be written

$$u = e^{i\mu x}(A_1 \cos a\mu t + B_1 \sin a\mu t).$$

Similarly,

$$u = e^{-i\mu x}(A_2 \cos a\mu t + B_2 \sin a\mu t)$$

is a solution, and so also is

$$u = e^{i\mu x}(A_1 \cos a\mu t + B_1 \sin a\mu t) + e^{-i\mu x}(A_2 \cos a\mu t + B_2 \sin a\mu t).$$

As before, the latter may be written

$$u = A \cos \mu x \cos a\mu t + B \sin \mu x \cos a\mu t + C \sin \mu x \sin a\mu t + D \cos \mu x \sin a\mu t,$$

where A, B, C, D are arbitrary constants.

Since u must vanish, both when $t = 0$ and when $x = 0$, it follows that $A = B = D = 0$, and the solution will be the sum of any number of terms of the form

$$u = C \sin \mu x \sin a\mu t,$$

where C and μ are still arbitrary constants.

---

* This equation holds for the transverse vibrations of a stretched string. Its general solution we found (Ex. 1, §XIV·2) to be

$$u = \phi(x + at) + \psi(x - at),$$

which represents two waves traveling with speed a, one to the left, the other to the right. (For example, see Ames and Murnaghan, *Theoretical Mechanics*, §106.)

Exercise 2. $h^2 \dfrac{\partial^2\theta}{\partial x^2} - \dfrac{\partial\theta}{\partial t} = 0.$

This is the differential equation determining the temperature at any point in the plane $x =$ const., normal to the rectilinear flow of heat in a body whose exposed surface, the plane $x = 0$, undergoes simple-periodic variations of temperature.* The boundary condition is

$$\theta = \theta_0 \sin \omega t \quad \text{when} \quad x = 0.$$

If we put $\theta = ce^{ax+bt}$, the auxiliary equation is found to be

$$h^2a^2 - b = 0.$$

From the nature of the problem, b must be imaginary. Put $b = \pm i\alpha$, then $a = \pm\sqrt{\pm i\alpha}/h$. But

$$\pm i = \cos\frac{\pi}{2} \pm i \sin\frac{\pi}{2}.$$

Using De Moivre's theorem, we have

$$\sqrt{i} = \pm\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \frac{\pm\sqrt{2}(1+i)}{2}$$

and

$$\sqrt{-i} = \pm\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right] = \frac{\pm\sqrt{2}(1-i)}{2}.$$

$$\therefore \theta = ce^{\pm\sqrt{2\alpha}(1\pm i)x/2h\pm i\alpha t}$$
$$= ce^{\pm\sqrt{2\alpha}x/2h}\, e^{\pm(\sqrt{2\alpha}x/2h\pm\alpha t)i}.$$

Since θ does not grow indefinitely with x, we must select the minus sign for the first exponential factor $e^{-\sqrt{2\alpha}x/2h}$.

There is no loss of generality if we select the plus sign for the coefficient of αt in the other exponential factor. We still have a choice of sign for the third ± that appears. Using both and making two arbitrary choices of c, we can write

$$\theta = e^{-\sqrt{2\alpha}x/2h}[c_1e^{(\sqrt{2\alpha}x/2h+\alpha t)i} + c_2e^{-(\sqrt{2\alpha}x/2h+\alpha t)i}].$$

---

* This problem arises if an insulated rod, indefinite in length, is heated at one end periodically. It also arises in studying the behavior of the temperature at a point below the earth's surface. Thus, see Ingersoll and Zobel, *Mathematical Theory of Heat Conduction*, Chapter V.

As before, the terms in brackets may be replaced by real terms involving sine and cosine, giving us

$$\theta = e^{-\sqrt{2\alpha}x/2h}[A \sin(\sqrt{2\alpha}x/2h + \alpha t) + B \cos(\sqrt{2\alpha}x/2h + \alpha t)].$$

This will satisfy the initial condition if $A = \theta_0$, $B = 0$, and $\alpha = \omega$. Hence the desired form of the solution is

$$\theta = \theta_0 e^{-\sqrt{2\omega}x/2h} \sin(\sqrt{2\omega}x/2h + \omega t).$$

3. Find a solution of $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ which vanishes when $x = \infty$ and when $y = 0$.

4. Find a solution of $\partial^2 u/\partial x^2 - \partial^2 u/\partial y^2 = 0$ which vanishes when $x = \infty$ and when $y = \infty$; also one which vanishes when $x = \infty$ and becomes infinite when $y = \infty$.

XIV·9. Fourier series. In treatises on Analysis, it is proved that a function $f(x)$, which is single-valued, of bounded variation, and has only a finite number of maxima and minima in an interval 2π in length,* can be expressed uniquely as a Fourier series

$$(1)\quad f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots + a_n \cos nx + b_n \sin nx + \cdots.$$

This holds throughout the interval, except possibly for a finite number of values of the variable, depending upon the character of the function and upon the choice of the interval. If, for example, the interval is $(-\pi, \pi)$, the coefficients in the Fourier series are equal to

$$(2)\quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx,$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx,\ (n = 1, 2, \cdots).$$

If, in particular, $f(x)$ is an *odd* function, that is, if

$$f(-x) = -f(x),$$

---

* If the interval is not 2π in length, a unique expansion is still possible in terms of another variable. Thus, if the interval is of length a, the corresponding Fourier series involves sines and cosines of integral multiples of $y \equiv 2\pi x/a$.

it is not difficult to see* that

$$a_0 = a_1 = \cdots = a_n = \cdots = 0,$$

and the Fourier series involves sine terms only.

With this established, we can represent a given function $f(x)$, which is single-valued, of bounded variation, and has only a finite number of maxima and minima in the interval $(0, \pi)$, by a Fourier series which involves sines only and which holds for the interval $(0, \pi)$, except possibly for a finite number of values of x, including $x = 0$ and $x = \pi$. For, if we consider now a function $F(x)$ which is defined as

$$F(x) = f(x) \text{ in the interval } (0, \pi),$$
$$F(x) = -f(-x) \text{ in the interval } (-\pi, 0),$$

we have a function which coincides with the original function in the interval $(0, \pi)$ and is odd in the larger interval $(-\pi, \pi)$. Hence it contains only sine terms.

To calculate the numerical coefficients, we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \sin nx dx = -\frac{1}{\pi}\int_{-\pi}^{0} f(-x) \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx dx.$$

If we put $x = -y$, the first integral in the right-hand member becomes

$$-\frac{1}{\pi}\int_{\pi}^{0} f(y) \sin(-ny)(-dy) = -\frac{1}{\pi}\int_{\pi}^{0} f(y) \sin ny dy = \frac{1}{\pi}\int_{0}^{\pi} f(y) \sin ny dy.$$

This is equal to the second integral in the right-hand member. Hence,

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx dx. \tag{3}$$

---

* For, $\int_{-\pi}^{\pi} f(x) \cos nx dx = \int_{-\pi}^{0} f(x) \cos nx dx + \int_{0}^{\pi} f(x) \cos nx dx$. If we put $x = -y$ and note that $f(-y) = -f(y)$, the first of the integrals in the right-hand member becomes $\int_{\pi}^{0} f(y) \cos ny dy$, or $-\int_{0}^{\pi} f(y) \cos ny dy$. This is the negative of the second integral appearing in the right-hand member.

The series

$$(4) \qquad b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots,$$

where the b's are defined by (3), is known as the *Fourier half-range sine series.* It equals $f(x)$ for all values of x* in the interval $(0, \pi)$, except possibly for $x = 0$ and $x = \pi$. For, as is obvious from the form of the series (4), the latter takes the value 0 when $x = 0$ and $x = \pi$ for all choices of the function $f(x)$.

In an analogous manner, a function $f(x)$, satisfying the conditions mentioned before, can be developed as a Fourier half-range cosine series

$$(5) \qquad a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots,$$

where

$$(6) \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx, \; a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nxdx, \; (n = 1, 2, \cdots).$$

To show this, construct an *even* function $F(x)$, defined as

$$F(x) = f(x) \text{ in the interval } (0, \pi),$$
$$F(x) = f(-x) \text{ in the interval } (-\pi, 0),$$

and proceed in a manner similar to that employed in the case of the half-range sine series.

***Exercise* 1.** Expand $f(x) = x$ as a half-range sine series.

Here

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nxdx = \frac{2}{\pi}\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = (-1)^{n+1}\frac{2}{n}.$$

$$\therefore x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right).$$

This holds for $x = 0$, but fails for $x = \pi$.

***Exercise* 2.** Expand $f(x) = x$ as a half-range cosine series.

Using (6), we find

$$a_0 = \frac{\pi}{2}, \; a_n = 0 \text{ for } n \text{ even, and } -\frac{4}{\pi n^2} \text{ for } n \text{ odd.}$$

$$\therefore x = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right).$$

---

* Provided $f(x)$ is also continuous in the interval.

Exercise 3. Expand $f(x)$ as a half-range sine series if

$$f(x) = x \text{ in } (0, \pi/2) \quad \text{and} \quad f(x) = \pi - x \text{ in } (\pi/2, \pi).$$

Here

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} x \sin nx dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x) \sin nx dx$$

$$= \frac{2}{\pi}\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi/2} + \frac{2}{\pi}\left[-\frac{(\pi - x) \cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi}$$

$$= (-1)^{\frac{(n-1)}{2}} \frac{4}{\pi n^2} \text{ for } n \text{ odd, and } 0 \text{ for } n \text{ even.}$$

$$\therefore f(x) = \frac{4}{\pi}\left(\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right).$$

Exercise 4. Expand the following functions as half-range sine series for the interval $0 < x < \pi$:

(*a*) $f(x) = 1$; (*b*) $f(x) = 0$ from $x = 0$ to $\pi/2$ and $f(x) = \cos x$ from $x = \pi/2$ to π; (*c*) $f(x) = \cos x$; (*d*) $f(x) = e^x$.

XIV·10. Application of Fourier series. Use is frequently made of the expansion of a function by a Fourier series when, to cite a simple case, among the initial or boundary conditions there is one giving the dependent variable as a known function of one of the independent variables when the other independent variable is equal to zero. This function may be a constant, but it cannot be zero. The following exercises will illustrate:

Exercise 1. Consider the steady flow of heat along a thin metal strip, infinite in length, whose two surfaces are insulated. Suppose that the strip is of width a, that the temperature along the long edges and at infinity is constantly 0°, while at the short edge it is maintained at θ_0 degrees.

If the short edge is taken as the axis of x and one of the long edges as the axis of y, the differential equation to be solved is

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0,$$

and the initial or boundary conditions are

$$\theta(0, y) = \theta(a, y) = 0, \theta(x, \infty) = 0, \theta(x, 0) = \theta_0.$$

Using the method of §XIV·6, we put $\theta = ce^{\alpha x+\beta y}$. The auxiliary equation is

$$\alpha^2 + \beta^2 = 0.$$

Hence the sum of any number of terms of the form

$$(1) \qquad c_1e^{\beta y}e^{i\beta x} + c_2e^{\beta y}e^{-i\beta x}, \quad \text{or} \quad e^{\beta y}(c_1e^{i\beta x} + c_2e^{-i\beta x}),$$

will satisfy the differential equation. As we have shown before (§VI·4), the expression (1) may be written

$$(2) \qquad Ae^{\beta y} \sin \beta x + Be^{\beta y} \cos \beta x,$$

where A, B, β are arbitrary constants.

Since $\theta(0, y) = 0$, we must make $B = 0$.

Since $\theta(a, y) = 0$, we must make $\beta = n\pi/a$, where n is an integer.

Since $\theta(x, \infty) = 0$, β must be negative, viz. $-n\pi/a$, where n is now any positive integer. Hence our expression $\theta(x, y)$ must consist of a sum of terms of the form

$$(3) \qquad \theta = \sum_n A_n e^{-n\pi y/a} \sin \frac{n\pi x}{a}.$$

Finally, we must select the A's so that $\theta(x, 0) = \theta_0$, i.e.,

$$(4) \quad \theta_0 = A_1 \sin \frac{\pi x}{a} + A_2 \sin \frac{2\pi x}{a} + \cdots + A_n \sin \frac{n\pi x}{a} + \cdots$$

for $0 < x < a$. This is a Fourier half-range sine series, in which the variable is $\pi x/a$.

In Ex. 4 of §XIV·9, it is found that

$$1 = \frac{4}{\pi}(\sin z + \tfrac{1}{3} \sin 3z + \tfrac{1}{5} \sin 5z + \cdots);$$

and this holds for $0 < z < \pi$. Putting $z = \pi x/a$, we have

$$(5) \qquad 1 = \frac{4}{\pi}\left(\sin \frac{\pi x}{a} + \tfrac{1}{3} \sin \frac{3\pi x}{a} + \tfrac{1}{5} \sin \frac{5\pi x}{a} + \cdots\right),$$

which holds for $0 < x < a$. Multiplying both members of (5) by θ_0, we have the values of the A's in (4). Hence the desired solution of our problem is

$$\theta = \frac{4\theta_0}{\pi}\left[e^{-\pi y/a} \sin \frac{\pi x}{a} + \frac{e^{-3\pi y/a}}{3} \sin \frac{3\pi x}{a} + \frac{e^{-5\pi y/a}}{5} \sin \frac{5\pi x}{a} + \cdots\right].$$

***Exercise* 2.** Suppose that a thin metal rod of length a is heated so that the temperature at a point, at a distance x from one end of the rod, is $f(x)$, a known function of x. The rod is then instantaneously insulated completely as to the passage of heat, and the ends are thoroughly packed in ice. It is desired to find $\theta(x, t)$, the temperature at each point, as a function of x and the time t, the latter being 0 when the ends are packed.

The differential equation for the flow of heat in this case is

$$h^2 \frac{\partial^2\theta}{\partial x^2} - \frac{\partial\theta}{\partial t} = 0;$$

and the initial or boundary conditions are

$$\theta(0, t) = \theta(a, t) = 0,\ \theta(x, \infty) = 0,\ \theta(x, 0) = f(x).$$

To make the problem precise, we shall suppose

$$f(x) = x \text{ for } 0 \leq x \leq \frac{a}{2}, \quad \text{and} \quad f(x) = a - x \text{ for } \frac{a}{2} \leq x \leq a.$$

In order to solve the equation, we shall let $\theta = ce^{\alpha x+\beta t}$. The auxiliary equation is

$$h^2\alpha^2 - \beta = 0.$$

Hence the sum of any number of terms of the form

$$ce^{\alpha x+h^2\alpha^2 t} \tag{6}$$

will satisfy the differential equation. Here c and α are arbitrary constants.

Since $\theta(x, \infty) = 0$, it is clear that α^2 must be negative. Let it be equal to $-p^2$. Then $\alpha = \pm ip$.

Taking the sum of a pair of terms of the form (6), we have

$$e^{-h^2p^2t}(c_1e^{ipx} + c_2\ {}^{-ipx}).$$

We can write this, as before, in the form

$$e^{-h^2p^2t}(A \sin px + B \cos px),$$

where A, B, p are arbitrary constants.

Since $\theta(0, t) = 0$, we must make $B = 0$.

Since $\theta(a, t) = 0$, we must make $p = n\pi/a$, where n is an integer. There is no restriction in supposing n positive. Hence our expression for $\theta(x, t)$ must consist of a sum of terms of the form

$$\theta = \sum_n A_n e^{-h^2n^2\pi^2t/a^2} \sin \frac{n\pi x}{a}.$$

Finally, we must select the A's so that $\theta(x, 0) = f(x)$, i.e.,

$$f(x) = \sum_n A_n \sin \frac{n\pi x}{a} \text{ for } 0 \leq x \leq a.$$

This is a Fourier half-range sine series, in which the variable is $\pi x/a$.

In our exercise, $f(x)$ was defined thus:

$$f(x) = x \quad \text{for} \quad 0 \leq x \leq a/2, f(x) = a - x \quad \text{for} \quad a/2 \leq x \leq a.$$

Putting $z = \pi x/a$, or $x = az/\pi$, we have

$$\begin{aligned} f(az/\pi) = F(z) &= az/\pi \quad \text{for} \quad 0 \leq z \leq \pi/2, \\ F(z) &= (\pi - z)a/\pi \quad \text{for} \quad \pi/2 \leq z \leq \pi. \end{aligned}$$

Comparing this with Ex. 3 of §XIV·9, we see that the Fourier half-range sine series for this function is

$$F(z) = \frac{4a}{\pi^2}\left(\sin z - \frac{\sin 3z}{3^2} + \frac{\sin 5z}{5^2} - \cdots\right);$$

and this holds for $0 \leq z \leq \pi$. Hence the desired solution of our problem is

$$\theta = \frac{4a}{\pi^2}\left[e^{-h^2\pi^2 t/a^2} \sin \frac{\pi x}{a} - \frac{e^{-9h^2\pi^2 t/a^2}}{9} \sin \frac{3\pi x}{a} + \frac{e^{-25h^2\pi^2 t/a^2}}{25} \sin \frac{5\pi x}{a} - \cdots\right].$$

This holds for $0 \leq x \leq a$.

Exercise **3.** Find the solution for the problem in Ex. 2, in case $f(x) = 100°$, a constant. This can be realized by immersing the rod originally in boiling water.

XIV·11. Partial differential equation of the second order, linear in the second derivatives. Monge's method. The general type of a partial differential equation of the second order, linear in the second derivatives is

$$Rr + Ss + Tt = V, \tag{1}$$

where R, S, T, V are functions of x, y, z, p, q. Gaspard Monge (1746–1818) suggested a method, which is known by his name, by which a first or intermediary integral is found in the form of a partial differential equation of the first order involving an arbitrary function. The solution of this equation by any of the methods of Chapter XIII, or otherwise, will then give the general solution. While this method applies only in case R, S, T, V satisfy certain conditions, differential equations of this type occur sufficiently often to justify our giving here at least the rule by which solutions are obtained by this method.*

In addition to

$$dz = pdx + qdy, \tag{2}$$

we have

$$\begin{cases} dp = rdx + sdy, \\ dq = sdx + tdy. \end{cases} \tag{3}$$

Eliminating r and t from (1) and (3), we have

$$s(Rdy^2 - Sdxdy + Tdx^2) - (Rdydp + Tdxdq - Vdxdy) = 0. \tag{4}$$

Whenever it is possible to satisfy simultaneously

$$Rdy^2 - Sdxdy + Tdx^2 = 0 \tag{5}$$

and

$$Rdydp + Tdxdq - Vdxdy = 0,\dagger \tag{6}$$

equation (4) will be satisfied and, therefore, so will (1). Equation (5) is equivalent to two equations of the first order,

$$dy - W_1(x, y, z, p, q)dx = 0 \quad \text{and} \quad dy - W_2(x, y, z, p, q)dx = 0, \tag{7}$$

which become identical in case

$$4RT = S^2. \tag{8}$$

Equations (2) and (6), together with either one of (7), constitute a system of three total differential equations in the five variables x, y, z, p, q. Such a system can be solved only in case certain conditions are fulfilled, and it is for this reason that Monge's method is not always applicable. It can be employed, if it is possible to find

$$u_1(x, y, z, p, q) = c_1, \quad u_2(x, y, z, p, q) = c_2,$$

---

* For a detailed account of this method, see Forsyth, *A Treatise on Differential Equations*, §229 and following. For a more recent treatment of the subject, see Goursat, *Sur l'intégration des équations aux dérivées partielles du second ordre.*

† Equations (5) and (6) are usually referred to as *Monge's equations.*

two independent solutions of this system. In this case, it can be shown that

$$u_1 = \phi(u_2), \tag{9}$$

where ϕ is an arbitrary function, is an *intermediary* (or *intermediate*) *integral.* Looked upon as a partial differential equation of the first order, (9) must be integrated again. Its general solution will be the solution of (1).

If it happens that not only one equation of (7), but each one, together with (2) and (6), determines a system that can be solved, we have two intermediary integrals (9). Solving these for p and q, we put the values of the latter in $dz = pdx + qdy$. The integral of this is the general solution of (1).

***Exercise* 1.** $q^2r - 2pqs + p^2t = 0.$

Monge's equations are

$$q^2dy^2 + 2pqdxdy + p^2dx^2 = 0,$$
$$q^2dydp + p^2dxdq = 0.$$

The first of these equations is a perfect square,

$$(qdy + pdx)^2 = 0.$$

Substituting this in the second equation, we find

$$qdp - pdq = 0;$$

whence

$$\frac{p}{q} = c_1.$$

The first of the Monge equations, combined with $dz = pdx + qdy$, gives

$$dz = 0;$$

whence

$$z = c_2.$$

An intermediary integral is, therefore,

$$p = q\phi(z).$$

In this case, we have only one intermediary integral. Hence we must integrate it. Since it is linear, it can be solved by the method of Lagrange (§XIII·1). Its general solution is

$$x\phi(z) + y = \psi(z).$$

Exercise **2.** $r - a^2t = 0.$*

Monge's equations are

$$dy^2 - a^2dx^2 = 0, \quad \text{or} \quad dy - adx = 0 \quad \text{and} \quad dy + adx = 0,$$
$$dydp - a^2dxdq = 0.$$

Using $dy - adx = 0$, we have $y - ax = c_1$. Combining it with the second of Monge's equations, we obtain

$$dp - adq = 0; \quad \text{whence} \quad p - aq = c_2.$$

Hence an intermediary integral is

$$p - aq = \psi(y - ax).$$

Using the other equation, $dy + adx = 0$, we find a second intermediary integral

$$p + aq = \phi(y + ax).$$

Solving these for p and q, we have

$$p = \tfrac{1}{2}[\phi(y + ax) + \psi(y - ax)],$$
$$q = \frac{1}{2a}[\phi(y + ax) - \psi(y - ax)].$$

We must now solve

$$dz = \tfrac{1}{2}[\phi(y + ax) + \psi(y - ax)]dx + \frac{1}{2a}[\phi(y + ax) - \psi(y - ax)]dy$$
$$= \frac{1}{2a}\phi(y + ax)(dy + adx) - \frac{1}{2a}\psi(y - ax)(dy - adx),$$

which is exact. Since ϕ and ψ are symbols of arbitrary functions, we shall retain them and write the solution in the form

$$z = \phi(y + ax) + \psi(y - ax).$$

There is no loss in failing to add an arbitrary constant, since either of the arbitrary functions may be supposed to include it.

---

* A much simpler method of solution for this equation was given in §XIV·2. This equation plays an important rôle in Mathematical Physics. It was first integrated by Jean-le-Rond D'Alembert (1717–1783) in a memoir entitled *Récherches sur les vibrations des cordes sonores*, presented in 1747 to the Berlin Academy. In studying the vibrations of a stretched elastic string, he considered the equation in the form $\partial^2y/\partial t^2 - \partial^2y/\partial x^2 = 0$, where t is the time, and x and y are the rectangular coördinates of a point of the string, x the coördinate measured along the line joining the extremities of the string, and y the displacement of the point from the position of equilibrium. His proof is given in Marie, *Histoire des Sciences Mathématiques et Physiques*, t. VIII, p. 217.

Exercise **3.** $r - t = -\dfrac{4p}{x+y}$.

Monge's equations are

$$dy^2 - dx^2 = 0, \quad \text{or} \quad dy - dx = 0 \quad \text{and} \quad dy + dx = 0,$$

$$dydp - dxdq + \frac{4p}{x+y}dxdy = 0.$$

Using $dy - dx = 0$, we have $y - x = c_1$. Combining it with the second of Monge's equations, we have

(10) $$2xdp + 4pdx - 2xdq + c_1(dp - dq) = 0.$$

Also,

$$dz = pdx + qdy$$

becomes

$$dz = pdx + qdx.$$

Combining this with (10), we obtain

$$2(xdp + pdx) - 2(xdq + qdx) + c_1(dp - dq) + 2dz = 0.$$

This is exact, and has for solution

$$(2x + c_1)(p - q) + 2z = c_2,$$

or

$$(x + y)(p - q) + 2z = c_2.$$

Hence an intermediary integral is

$$(x + y)(p - q) + 2z = \phi(y - x).$$

Using the equation $dy + dx = 0$, we have a system of total differential equations which is not integrable. Hence, we must integrate the intermediary integral. This is linear. Employing Lagrange's method, we pass to

$$\frac{dx}{x+y} = \frac{-dy}{x+y} = \frac{dz}{\phi(y-x) - 2z}.$$

From the equation of the first two members we have

$$x + y = a.$$

Replacing y by its value $a - x$, we have

$$\frac{dx}{a} = \frac{dz}{\phi(a - 2x) - 2z},$$

or

$$\frac{dz}{dx} + \frac{2}{a}z = \frac{1}{a}\phi(a - 2x).$$

This is a linear ordinary differential equation of the first order. An integrating factor (§II·9) is $e^{2x/a}$, and the solution is

$$aze^{2x/a} = \int e^{2x/a}\phi(a - 2x)dx + b.$$

Replacing a by its value $x + y$, we have the general solution

$$(x + y)ze^{2x/(x+y)} - \int e^{2x/a}\phi(a - 2x)dx = \psi(x + y).$$

Here, as in the case of the non-linear partial differential equation of the first order (§XIII·6), the general solution cannot be obtained explicitly. For, until the form of ϕ is known, the above integral cannot be calculated. In any given case, in which the initial conditions determine ϕ, the a which appears in the integral must be replaced by $x + y$ after the integration has been effected.

4. $q(1 + q)r - (p + q + 2pq)s + p(1 + p)t = 0.$
5. $ps - qr = 0.$
6. $(b + cq)^2r - 2(b + cq)(a + cp)s + (a + cp)^2t = 0.$

XIV·12. Special method. At times, by considering one or the other of the independent variables as a constant temporarily, the equation may be looked upon as an ordinary differential equation. Of course, an arbitrary function of the variable considered as constant must take the place of the arbitrary constant in the solution. The following exercises will illustrate this:

***Exercise* 1.** $xr = p.$

Letting y be a constant temporarily, we can write this as the ordinary differential equation

$$x\frac{dp}{dx} = p, \quad \text{or} \quad \frac{dp}{p} = \frac{dx}{x}.$$

Integrating, we have $p = xf(y)$, where $f(y)$ is an arbitrary function. Again, letting y be a constant, we have

$$\frac{dz}{dx} = xf(y);$$

whence $z = x^2f(y) + \phi(y)$, where $\phi(y)$ is another arbitrary function. Here the factor $\frac{1}{2}$, arising on the right, is included in $f(y)$.

Exercise **2.** $r + s + p = 0$.

If we integrate this, considering y as a constant, we have

$$p + q + z = f(y).$$

This is linear and of the first order. Employing Lagrange's method, we pass to

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(y) - z}.$$

From the first two members we find

$$x - y = a.$$

From the last two members we have the linear ordinary differential equation

$$\frac{dz}{dy} + z = f(y).$$

An integrating factor is e^y (§II·9), and the solution is

$$\begin{aligned} ze^y &= \int f(y)e^y dy + b \\ &= \phi(y) + b. \end{aligned}$$

Hence the general solution is

$$ze^y - \phi(y) = \psi(x - y),$$

or

$$z = \phi(y) + e^{-y}\psi(x - y),$$

where the factor e^{-y} is included in $\phi(y)$.

3. $yt - q = xy^2$. **4.** $s = xy$. **5.** $r + p = xy$.

XIV·13. Summary. The number of classes of partial differential equations of higher order than the first which can be integrated by elementary means is very small. In this chapter we have dealt almost entirely either with differential equations linear in the dependent variable and all of its derivatives, or with differential equations of the second order linear in the highest derivatives only. The latter class occasionally yields to Monge's method, §XIV·11.

If the equation is linear in the dependent variable and all of its derivatives, and has constant coefficients, the general method of §XIV·6 applies.

If the linear equation with constant coefficients is "homogeneous," that is, if the dependent variable is absent and all the derivatives that appear are of the same order, the method of §XIV·2 applies.

If the equation is linear, but the coefficients are not constants, a transformation can sometimes be found to reduce the equation to one with constant coefficients (§XIV·7).

At times the special method of §XIV·12 can be applied directly to an equation.

Exercise **1.** $ys = x + y.$

2. $r - s - 6t = xy.$ **3.** $zr + p^2 = 3xy^2.$

4. $xr - (x + y)s + yt = \dfrac{x+y}{x-y}(p - q).$

[Use Monge's method. Also, let $x + y = u$, $xy = v$.]

5. $xr - p = xy.$ **6.** $r - t - 3p + 3q = e^{x+2y}.$

7. $x^2r - y^2t = (x + 1)y.$

8. $x^2r + 2xys + y^2t + xp + yq - n^2z = 0.$

9. $xr + p = 9x^2y^2.$ **10.** $s - t = x/y^2.$

11. $s = pq.$ **12.** $r - 2s + t = (x + y)^2.$

13. $(q + 1)^2r - 2(pq + p + q + 1)s + (p + 1)^2t = 0.$

14. $qs - pt = 0.$ **15.** $s + p - q = z + xy.$

NOTE I

Condition that a relation exist between two functions of two variables. *If u and v are two functions of x and y, the necessary and sufficient condition that an identical* relation exist between them is that their functional determinant* (also called their *Jacobian*) *vanishes identically, that is, for all values of x and y*

$$\frac{\partial(u, v)}{\partial(x, y)} = J_{x,y}(u, v) = (u, v)_{x,y} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0.$$

1° To prove the necessity of the condition:

If $u = \phi(v)$, we obtain, on differentiating,

$$\frac{\partial u}{\partial x} = \frac{d\phi}{dv}\frac{\partial v}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{d\phi}{dv}\frac{\partial v}{\partial y}.$$

These two equations in the single quantity $d\phi/dv$ can hold simultaneously only if

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0,$$

which proves the necessity of the condition.

---

* By an *identical relation* between u and v is meant one that holds for all values of x and y. Thus, if $u = x + y$ and $v = (x + y)^2$, $u^2 = v$ for all values of x and y. If there is no identical relation existing between u and v, we may still equate one of them to a function of the other. But the resulting relation is not an identity; it holds only for selected pairs of values of x and y. Thus, if $u = x + y$ and $v = x - y$, the relation $u^2 = v$ holds only for the coördinates of the points on the parabola $(x + y)^2 = x - y$.

2° Now, to prove the sufficiency of the condition:

Suppose u and v to be given by

$$u = f_1(x, y),\ v = f_2(x, y).$$

From these we can eliminate y. The resulting relation, when solved for u, may be written as

$$u = \phi(x, v).$$

Since x and y are independent variables, we obtain, upon differentiating with respect to each of them in turn,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x}, \\ \frac{\partial u}{\partial y} &= \phantom{\frac{\partial \phi}{\partial x} + {}} \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y}. \end{aligned}$$

If $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$, these equations can hold simultaneously only provided $\frac{\partial \phi}{\partial x} = 0$. But this means that ϕ is free of x. Hence, when the Jacobian vanishes,

$$u = \phi(v),$$

which proves the sufficiency of the condition.

Remark. This theorem can be extended to n functions of n independent variables.

NOTE II

Condition that n functions of a single variable should satisfy a linear identical relation with constant coefficients. Consider, first, the case of two functions $y_1(x)$ and $y_2(x)$. Suppose that an identity* of the form

$$a_1y_1 + a_2y_2 = 0 \tag{1}$$

---

* By an *identity* we mean an equation which holds for all values of the variable.

exists, where a_1 and a_2 are constants, both of which are not zero. Differentiating, we have

$$a_1y_1' + a_2y_2' = 0. \tag{2}$$

In order that these two homogeneous relations in a_1 and a_2 may hold simultaneously, it is necessary that

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0, \tag{3}$$

for all values of x. Hence (3) is a necessary condition.

To show that (3) is also sufficient, we write it in the expanded form

$$y_1y_2' - y_2y_1' = 0,$$

and integrate, obtaining the linear relation with constant coefficients

$$y_2 = cy_1.$$

Consider, now, the case of three functions $y_1(x)$, $y_2(x)$, $y_3(x)$. Suppose that an identity of the form

$$a_1y_1 + a_2y_2 + a_3y_3 = 0 \tag{4}$$

exists, where a_1, a_2, a_3 are constants and all of them are not zero. Differentiating this twice, we have

$$\begin{cases} a_1y_1' + a_2y_2' + a_3y_3' = 0, \\ a_1y_1'' + a_2y_2'' + a_3y_3'' = 0. \end{cases} \tag{5}$$

In order that these three relations, homogeneous in a_1, a_2, a_3, may be satisfied simultaneously, it is necessary that

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = 0. \tag{6}$$

Hence (6) is a necessary condition that a linear identical relation with constant coefficients exist between the functions y_1, y_2, y_3. To show that it is also sufficient, we proceed as follows:

If a determinant vanishes, there is an identical linear relation (not necessarily with constant coefficients) connecting corresponding elements of the rows (and of the columns). Thus, since (6) vanishes, three quantities μ_1, μ_2, μ_3, all of which are not zero, exist such that

$$(7)\qquad \begin{cases} \mu_1 y_1 + \mu_2 y_2 + \mu_3 y_3 = 0, \\ \mu_1 y_1' + \mu_2 y_2' + \mu_3 y_3' = 0, \\ \mu_1 y_1'' + \mu_2 y_2'' + \mu_3 y_3'' = 0. \end{cases}$$

For example, we can take

$$(8)\quad \mu_1 : \mu_2 : \mu_3 = \left\| \begin{matrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \end{matrix} \right\| = \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix} : \begin{vmatrix} y_3 & y_1 \\ y_3' & y_1' \end{vmatrix} : \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

If any one of these two-rowed determinants is zero, a linear identity of the form (4) results. Thus, for example, suppose that the first two-rowed determinant is zero. We saw above that this is the condition for the existence of a linear identical relation connecting y_2 and y_3, viz.,

$$y_2 - c y_3 = 0.$$

Moreover, this is of the form (4), where $a_1 = 0$, $a_2 = 1$, $a_3 = -c$.

If none of these two-rowed determinants vanishes, none of μ_1, μ_2, μ_3 is zero. Moreover, we shall show that they are proportional to constants, so that the first identity of (7) is of the form (4).

Differentiating the first two equations of (7), and taking account of the last two, we have

$$(9)\qquad \begin{aligned} \mu_1' y_1 + \mu_2' y_2 + \mu_3' y_3 &= 0, \\ \mu_1' y_1' + \mu_2' y_2' + \mu_3' y_3' &= 0. \end{aligned}$$

Since these are homogeneous in μ_1', μ_2', μ_3', we have

$$(10)\ \mu_1' : \mu_2' : \mu_3' = \left\| \begin{matrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \end{matrix} \right\| = \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix} : \begin{vmatrix} y_3 & y_1 \\ y_3' & y_1' \end{vmatrix} : \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Comparing these with (8), we have

$$(11)\qquad \frac{\mu_1'}{\mu} = \frac{\mu_2'}{\mu_2} = \frac{\mu_3'}{\mu_3} = \lambda(x),$$

where, in general, $\lambda(x)$ involves x, although in special cases it may be a constant. Integrating each of the differential equations (11) in turn, we have

$$\mu_1 = c_1 e^{\int \lambda(x)dx}, \quad \mu_2 = c_2 e^{\int \lambda(x)dx}, \quad \mu_3 = c_3 e^{\int \lambda(x)dx};$$

i.e.,

$$\mu_1 : \mu_2 : \mu_3 = c_1 : c_2 : c_3, \text{ a set of constants.}$$

The sufficiency of the condition (6) is thus established.

By induction, the method can be employed to show that *the necessary and sufficient condition that* $y_1(x), y_2(x), \cdot\cdot\cdot, y_n(x)$, *a set of n functions of a single variable, should satisfy a linear identical relation with constant coefficients*

$$a_1y_1 + a_2y_2 + \cdot\cdot\cdot + a_ny_n = 0$$

in which not all the coefficients are zero, is

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0. \tag{12}$$

Hence the necessary and sufficient condition that the n functions $y_1(x), y_2(x), \cdot\cdot\cdot, y_n(x)$ *be linearly independent is that the determinant in* (12) *does not vanish identically.*

The determinant in (12) is known as the Wronskian of $y_1, y_2, \cdot\cdot\cdot, y_n$. It is named for J. Hoëné-Wronski (1778–1853).

NOTE III

General summary. The following is an index to the various methods, given in this treatise, for solving differential equations:

1° *Single ordinary differential equation*

(*a*) of first order

general solution, Chapters II, IV, IX,
singular solution, Chapter V,

(*b*) of higher order
    linear with constant coefficients, Chapter VI,
        of Cauchy type, §VI·11,
        with variable coefficients, Chapters VII, VIII, IX, §XI·6,
    non-linear, Chapters VIII, IX, §XI·6.

2° *Single partial differential equation*

(*a*) of first order, Chapter XIII,
(*b*) of higher order, Chapter XIV.

3° *Single total differential equation*, Chapter X.

4° *System of differential equations*
    ordinary or total, Chapter XI.

INDEX

The numbers refer to pages.

The following abbreviations are used: c.c. = constant coefficients, d.e. = differential equation, l. = linear, o. = ordinary, p. = partial, t. = total.

d'Alembert, 103, 297.
Ames and Murnaghan, 286.
Angle between two curves, 55.
Applications to
  Chemistry, 66–68.
  Electricity, 63, 139.
  Geometry, 46–60, 80, 85, 172, 236, 273, 274.
  Physics, 60–63, 65, 68–73, 85, 128–139, 141–147, 173, 240, 241, 286, 287, 291–294.
Arbitrary constant, 5.
Auxiliary equation, 104, 276, 282.
  roots of, complex, 105, 279.
  roots of, repeated, 104, 277.

Bateman, 95, 241.
Bernoulli, 32.
Bernoulli's equation, 32.
Bessel, 205.
Bessel's equation, 205.
Bessel's functions, 205, 206.
Boole, 112.
Boundary condition, 51.

Cantilever beam, 70.
  neutral surface of, 70.
Cauchy, 124.
Cauchy's l.d.e., 124.
Cayley, 99.
c-discriminant relation, 89.
  geometrical significance, 89.
Center of force, 240.
Central force, 141.
Characteristic equation, 104.
Charpit, 263.
  method of Lagrange and, 263.
Chrystal, 99.
Clairaut, 35, 80.
Clairaut's equation, 80.
  extended, 272.
Classification of differential equations, 3.
Coble, 122.
Commutative operations, 109.
Complementary function, 103, 276.
Complete l.d.e., 101.
Complete solution of o.d.e., 9.
  of p.d.e., 261, 262.
Condition for exactness
  of o.d.e. of first order, 24.
  of l.o.d.e. of higher order, 163.
Condition for integrability of t.d.e., 211.
  for l. independence of functions, 303.
  for relation between functions, 302.
  for repeated roots of an algebraic equation, 88.
Curve of pursuit, 173.
Cuspidal locus, 93.

Damped vibration, 130.
Darboux, 99.
Degree of a d.e., 3.
Dependent variable absent, 155.
Derivation of an o.d.e., 6.
  of a p.d.e., 242, 245.
Differential equation, 1, 7.

Differential equation
  of a family of curves, 46, 48.
  of simple harmonic motion, 129.
Differential expressions in Geometry, formulas for, 52, 53, 274.
Discriminant, 87.
  relation, 89.
Distribution of temperature, 134, 136, 137.
Double root, condition for, 88.

Electric circuit, 63, 139.
Envelope, 86, 260, 261.
Euler, 35, 103, 200, 206.
  factor or multiplier, 35.
Even function, 290.
Exact differential, 23.
  condition for, 24.
Exact d.e., 23, 27, 162.
Existence theorem for o.d.e., 175, 178.
  for l.o.d.e., 182.
  for p.d.e., 249.
Extraneous loci, 92.

Falling body, 60, 61.
Family of curves, 48.
  equation of, 48.
Fine, 99.
First integral, 162.
Flow of heat, 65, 287, 291, 292.
Forced vibration, 132.
Forsyth, 206, 295.
Fourier series, 288.
  half-range sine series, 290.
Frobenius, 184.
Fuchs, 184.

Gauss, 200, 203.
Gauss's equation, 199.
General integral, 102, 103.
General solution of o.d.e., 9.
  of p.d.e., 248, 252, 262.
General summary, 306.
Geometrical interpretation of
  discriminant relations, 92.
  existence theorems, 179, 249.
Geometrical interpretation of
  o.d.e. of first order, 46.
  p.d.e. of first order, 259.
  system of o.d.e. of first order, 235.
  t.d.e., 222.
Glaisher, 99.
Goursat, 295.
Goursat-Bourlet, 249, 260.

Hill, 94, 99, 256, 261.
Homogeneous function, 15.
  Euler's theorem for, 34.
Homogeneous l.o.d.e., 101.
  l.p.d.e. of first order, 250.
  l.p.d.e. with c.c., 276.
  o.d.e., 15, 34, 158.
  t.d.e., 216.
Hypergeometric series, 199, 201.

Identical functional relation, 302.
  condition for, 302.
Identical linear relation, 303.
  condition for, 303.
Identity, 303.
Independent arbitrary constants, 8.
Independent integrating factors, 27.
Independent variable absent, 157.
Indicial equation, 187.
  roots of, differ by an integer, 197.
  roots of, repeated, 195.
Infinity of curves, 48.
Ingersoll and Zobel, 287.
Initial condition, 51.
Integrable t.d.e., 210.
  method of solution, 213, 215, 216, 219.
Integral, 3, 102.
  curve, 47.
  surface, 222.
Integrating a d.e., 10.
Integrating factor, 23, 34, 38, 39, 41, 164, 257.
  by inspection, 39.
  of o.d.e. of first order, indefinite in number, 24.
Integration in series, 175.
Integration term for term, 14.

Intermediary integral, 296.
Irreducible d.e. of first order, 74.
Isobaric function, 19.
Isobaric d.e., 19, 34, 159.
Isogonal trajectory, 55.

Jacobian, 302.

Kepler's laws, 142, 144.

Lacroix, 263.
Lagrange, 114, 250, 261, 263.
  method of, 250.
  and Charpit, method of, 263.
Legendre, 125, 203.
Legendre's equation, 125, 203.
Legendrian coefficient, 204.
Leibniz, 35.
Liebmann, 99.
Linear o.d.e., 30, 101.
  Cauchy's, 124.
  complete, 101.
  condition for exactness of, 163.
  d.e. of first order reducible to, 32.
  homogeneous, 101.
  Legendre's, 125, 203.
  of first order, 30.
  of second order, 148.
  reducible to one with c.c., 124.
  solution of, in series, 184.
  system of, with c.c., 228.
  with c.c., 103.
  with particular integral known, 149, 167.
Linear p.d.e., 246, 250, 275, 294.
  homogeneous, 250.
  homogeneous, with c.c., 276.
  non-homogeneous, with c.c., 281.
  of first order, 246, 250.
  of second or higher order, 275, 294.
  reducible to one with c.c., 284.
Linear relation, 303.
  condition for, 303.
Linearly independent functions, 102.
Lines of force due to two magnetic poles, 68.
Liouville, 165.
Literal constant, 5.
Lobatto, 112.

Marie, 297.
Monge, 294.
Monge's equations, 295.
Monge's method, 294.
Motion of a particle, 60–63, 141, 146, 173, 240, 241.

Nodal locus, 93.
Non-homogeneous l.p.d.e. with c.c., 281.
Non-integrable t.d.e., 221.
Non-linear p.d.e. of first order, 260.

Odd function, 288.
Order of a d.e., 3.
Ordinary d.e., 3.
Orthogonal trajectory, 56, 236.

Partial d.e., 3.
Particular curve, 95.
Particular integral of l.o.d.e., 102 108, 110, 112, 114, 118.
  of l.p.d.e., 280.
Particular solution of o.d.e., 9.
  of p.d.e., 261, 285.
Perry, 73.
Petrovitch, 95.
Piaggio, 95.
Picard, 99.
Primitive of o.d.e., 7.
  of p.d.e., 242, 245.

Quadrature, 10.

Rational, entire function, 88.
Rational operations of Algebra, 88.
Reducible d.e. of first order, 74.
Reduction of o.d.e. to equivalent system, 177, 237.
Region, 175.
Regular function, 178.
Resonance, 133.
Riccati, 206.

Riccati's equation, 206.
  analogy of, to l.d.e., 208.
Roots of auxiliary equation,
  complex, 105, 279.
  repeated, 104, 277.
Roots of indicial equation,
  differ by an integer, 197.
  repeated, 195.
Rothenberg, 100.

Schlesinger, 202.
Separation of variables, 12, 215, 270.
Series, integration in, 175.
Simple harmonic motion, 128.
  equation of, 129.
Simple pendulum, 128, 173.
Singular point, 183, 249.
Singular solution of o.d.e., 86, 89, 179.
  of p.d.e., 260.
Slope of a curve, 46.
Solution in series, 180.
  for l.o.d.e., 184.
Solution of a d.e., 3.
  complete, 9, 261, 262.
  general, 9, 248, 252, 262.
  particular, 9, 261.
  singular, 86, 89, 179, 260.
  special, 37, 255.
Solving a d.e., 10.
Summary, general, 306.
Symbol D, 101, 228, 275.
  $\mathfrak{D}$, 124, 161, 275.
Symbolic operator $(D - \alpha)$, 108.
System of differential equations, 226.
  general method of solution, 226.
  of l.o.d.e. with c.c., 228.
  of o.d.e. of the first order, 231.
  of t.d.e., 236.
  replacing an o.d.e., 177, 237.

Tac-locus, 94.
Tait and Steele, 61.
Total d.e., 210, 219.
  homogeneous, 216.
  integrable, 210, 219.
  method of solution, 213, 215, 216, 219.
  non-integrable, 221.
  system of, 236.
Trajectory, 55.

Undetermined coefficients, method of, 118.

Variables, separation of, 12, 215, 270.
Variation of parameters, 114.
Vertical column, 146.
Vibration of stretched string, 286, 297.

Wronski, 306.
Wronskian, 306.

y'-discriminant relation, 89.
  geometrical significance of, 92.

Zonal harmonic, 204, 205.

ANSWERS

ANSWERS

SECTION I·4

3. $y = xy' + \sqrt{1 - y'^2}.$

4. $r^2y''^2 = (1 + y'^2)^3.$

5. $xy'' - y' = 0.$

6. $2xyy' + x^2 - y^2 = 0.$

7. $xyy'' + xy'^2 - yy' = 0.$

8. $y'' - (k + l)y' + kly = 0.$

9. $xy'^2 - 2yy' - x = 0.$

10. $yy'^2 + 2xy' - y = 0.$

11. $y'^2 + y^2 = 1.$

12. $y = \sin \dfrac{xy'}{\sqrt{1 - y^2}}.$

SECTION II·1

4. $\sqrt{1 + x^2} + \log y = c.$

5. $x^2 + cy^2 = c - 1.$

6. $y = c \cos x - 3.$

7. $\cos x \cos y = c.$

8. $\sec x + \tan y = c.$

SECTION II·2

3. $\log x - x/y = c.$

4. $x^2 - 2cy + c^2 = 0.$

5. $(x^2 + y^2)y^2 = cx^6.$

6. $\log x - \sin \dfrac{y}{x} = c.$

7. $[x^2 + (1 - a)y^2]^a = cx^2.$

8. $x^2 + y^2 = c(my - x).$

SECTION II·3

2. $\tan^{-1}\left(\dfrac{y + 2}{2x + 2}\right) + \log [4(x + 1)^2 + (y + 2)^2] = c.$

3. $(x + y + 3)^3(x - y - 1) = c.$

4. $5x - 10y + \log (10x + 5y - 2) = c.$

5. $5x + 10y - \log (10x - 5y + 17) = c.$

6. $3x + y - 2 \log (5x + 2y + 1) = c.$

SECTION II·4

2. $x^2 + y = cxy,\ [m = 2].$

3. $x^2 + 2x^4y = c,\ [m = -2].$

4. $x^2y + x^3y^2 = c,\ [m = -1].$

5. $xy = c^2x + c,\ [m = -2].$

6. $cy^2 - x^2 = c^2,\ [m = \tfrac{1}{2}].$

SECTION II·5

1. $x + y = \tan(x + c)$. **2.** $x - y + a = \dfrac{e^{2x} - c}{e^{2x} + c}$.

3. $y = 2\log(2x + y) + c$. **4.** $\tan(x - y) = x + c$.

SECTION II·8

4. $x^2 + 2xy = c$. **5.** $3x^2 - 2xy + y^2 + x - 3y = c$.

6. $\log\sqrt{x^2 + y^2} + \tan^{-1}\dfrac{x}{y} = c$. **7.** $\sqrt{\dfrac{x}{y}} + \log y = c$.

8. $x^{5/2}y^{3/2} - x^{3/2}y^{7/2} = c$. **9.** $(x^2 - 1)y^{-3} - x^3 = c$.

10. $x^2 \sin\dfrac{y}{x} = c$.

SECTION II·9

2. $y \sin x = \log \sec x + c$. **3.** $y = x + c\sqrt{x^2 + 1}$,
4. $2y = (x + 1)^4 + c(x + 1)^2$. or $(x - y)^2 = c_1(x^2 + 1)$.
5. $y(1 + x^2)^2 = x^2 + 2\log x + c$. **6.** $y = x^2(1 + ce^{1/x})$.
7. $y = x^2 + c\sqrt{x^2 - 1}$, or $(x^2 - y)^2 = c_1(x^2 - 1)$.
8. $x = y^4 + cy^3$.

SECTION II·10

2. $\log y = x^2 + cx$. **3.** $y^2 = x^2 - 1 + ce^{-x^2}$.
4. $\cos y = 1 + ce^{-\cos x}$. **5.** $x^3y^4(e^x + c) = 1$.
6. $4y^{-3/2}(1+x+x^2)+3=c(1-x)^2$. **7.** $\sqrt{y + 1} = x + 1 + c\sqrt{x + 1}$.
8. $x^{-2} = 1 - y^2 + ce^{-y^2}$.

SECTION II·12

2. $x^2y + x^3y^2 = c$. **3.** $x^2y^2 + y = cx$.
4. $y^2 = cx$. [In addition, $x^3 = 3y^2$.]
5. $x^3 + y = cxy^2$. **6.** $(x^2y + 1)y^{1/2} = cx^{1/3}$.

SECTION II·13

1. $3x^4 + 8x^3y + 6x^2y^2 + 4x^2y = c$.
2. $x^2 - y^2 - cy = a^2$. **3.** $y^4 + xy^3 + 2x = cy^2$.

SECTION II·14

5. $\tan^{-1}\frac{y}{x} = x + c.$ **6.** $x^4 + 2(c-2)x^2 + 4y^2 + c^2 = 0.$

7. $x \log x + y^2 = cx.$ **8.** $2xy^2 - 1 = cx^2y^2.$

9. $x^2y^2 + y = cx.$ **10.** $x^2 + y^2 - cx = a^2.$

SECTION II·16

1. $x^2 + y = x \tan(x + c).$ **2.** $\log \frac{xy-1}{xy+1} = x^2 + c.$

4. $x^2 + y^2 = c(1 + x)^2.$

5. $(1 - x)^2y^2 + x^2 - \frac{2}{3}x^3 = c.$ **6.** $2y = x(1 + ce^{-x^2}).$

7. $\log[(x^2 - 2)^2 + (y^2 - 1)^2] + 2\tan^{-1}\frac{y^2 - 1}{x^2 - 2} = c.$

8. $4(x^2 + y)^3 = (2x^3 + 3xy + c)^2.$

SECTION II·17

1. $\sqrt{1 - x^2} + \sqrt{1 - y^2} = c$, or $(x^2 - y^2)^2 + 2c_1(x^2 + y^2) + c_1^2 - 4c_1 = 0.$

2. $\sin^{-1} x + \sin^{-1} y = c$, or $x^2 + y^2 + 2c_1xy = 1 - c_1^2.$

3. $x^3 + y + 3 = ce^{x^3/3}.$ **4.** $\frac{y - x + 1}{y - x - 1} = ce^{2y}.$

5. $x^3y^3(3e^x + c) = 1.$ **6.** $xy = ce^{x+y}.$

7. $\log y + \frac{x}{y} = c.$ **8.** $x^2 - 2cy - c^2 = 0.$

9. $\sin^{-1}\frac{y}{x} = \log x + c.$ **10.** $y = (x + a)^5 + c(x + a)^3.$

11. $x \sin\frac{y}{x} = c.$ **12.** $(x + y - 1)^3 = c(x - y + 3).$

13. $x^2 + y^2 - 2x + 2 = ce^{-x}.$ **14.** $x^4 - x^2 = cy^2.$

15. $y = x/(1 - x^2)^{1/2} + ce^{-x/(1-x^2)^{1/2}}.$ **16.** $(ay + 1)^2 = c(1 - x^2)y^2.$

17. $x^3y^2 - x^2 = cy.$ **18.** $x^{5/2}y^{3/2} - x^{3/2}y^{7/2} = c,$

19. $xy^4 + 4y = cx.$ or $x^3y^3(x - y^2)^2 = c_1.$

20. $y = \tan^{-1} x - 1 + ce^{-\tan^{-1} x}.$ **21.** $y = 2 \sin x - 2 + ce^{-\sin x}.$

22. $x^2y + 2x = cy.$

23. $x^3 - y^2 = cy.$ **24.** $3x^3y + y^3 = c.$

25. $x^2 + y^2 = c(x^2 + y^2)^{1/2}e^{\tan^{-1} y/x} + \frac{1}{2}(y - x).$

26. $x + y - 4 \log(2x + 3y + 7) = c.$

27. $xy(x^2 + y^2 - c) + 1 = 0.$ **28.** $x^2y^2(y^2 - x^2) = c.$

29. $\frac{1}{y} = \log x + 1 + cx.$ **30.** $(1 + x^2 + y^2)^{1/2} = \tan^{-1}\frac{y}{x} + c.$

31. $x^2y^2 = 2xy \log cy + 1.$ The equation is isobaric when $m = -1.$

32. $x = cye^{-2y}$. **33.** $y(2x^3y - 3)^2 = c$.

34. $x^3y^2 + (x^2 - 2x + 2)e^x = c$. **35.** $2y^{-2} = a(2x^2 + 1) + ce^{2x^2}$.

36. $ye^{\sqrt{x/y}} = c$. **37.** $x^2y^2 + x^3 = ce^x$.

38. $(x^2 + y^2)^2 + 2cx(x^2 + y^2) - c^2y^2 = 0$. In polar coördinates $\rho = c_1(1 - \cos\theta)$.

SECTION III·1

3. $xyy'^2 + (x^2 - y^2 - \lambda)y' - xy = 0$. **6.** $xy'^2 + (k - x - y)y' + y = 0$.

4. $yy'^2 + 2xy' - y = 0$. **7.** $2xy'^2 = (3x - 1)^2$.

5. $y = xy' + r\sqrt{1 + y'^2}$. **8.** $(2xy' - y)^2 = 8x^3$.

9. $y'' = 0$. This does not include the lines $x = c$, which are parallel to the y-axis. For these, $y' = \infty$. The differential equation of these ∞^1 lines is $dx/dy = 0$.

10. $x^2y''^2 - 2xy'y''(1 + y'^2) - (1 + y'^2)^2 = 0$.

11. $xyy'' + xy'^2 - yy' = 0$.

12. $3y'y''y^{\text{iv}} - 4y'y'''^2 - 3y''^2y''' = 0$.

13. (*a*) $y'y''' - 3y''^2 = 0$. (*b*) $y''' = 0$.

14. $r^2y''^2 - (1 + y'^2)^3 = 0$. **15.** $(1 + y'^2)y''' - 3y'y''^2 = 0$.

SECTION III·2

5. $xy = c$, equilateral hyperbolas.

6. $y^2 + (1 - k)x^2 = c$, central conics.

7. $\log\sqrt{x^2 + y^2} = \tan^{-1}\dfrac{y}{x} + c$, or $\rho = c_1e^{\theta}$, logarithmic spirals.

8. Same as **Ex. 7.** **9.** $\rho = ce^{\pm\theta}$.

10. $y^2 = 2kx + c$, parabolas. **11.** $y = ce^{-1/x}$.

12. $x^2 + y^2 - cy = 0$, circles through the origin, with centers on the y-axis. Also $y = c$.

13. $\rho(\theta + c) + k = 0$, reciprocal spirals.

14. $\rho = ce^{\theta/k}$, logarithmic spirals.

15. $\rho = ce^{\theta/\sqrt{k}}$.

16. $\rho = k(\theta + c)$, spirals of Archimedes.

17. $\rho = c\sin\theta$, same family of circles as in Ex. 12.

18. $\rho^2 = c\sin 2\theta$, lemniscates.

19. $\rho = c\sin^2(\theta/2)$, or $\rho = c_1(1 - \cos\theta)$, cardioids.

20. $\rho = 2c/(1 + \cos\theta)$, parabolas with foci at the origin.

SECTION III·3

5. $x^2 + y^2 - cx + a^2 = 0$. **6.** $2x^2 + y^2 = c$. **7.** $xy = c$.

8. $3x^2y + y^3 = c$. **9.** $y^a = cx^b$. **10.** $ay^{2-n} - bx^{2-n} = c$.

11. $(a - 2)x^2 - y^2 = cy^a$, $a \neq 2$; $\log y = c - x^2/y^2$, $a = 2$.

12. $\sqrt{\rho^2 - a^2} - a\cos^{-1}(a/\rho) = a\theta + c$, or
$\sqrt{x^2 + y^2 - a^2} - a\cos^{-1}(a/\sqrt{x^2 + y^2}) = a\tan^{-1}(y/x) + c.$
14. $x^2 + y^2 = c(my - x).$
15. $\rho = ce^{-m\theta}$, (compare Ex. 13).
17. $\rho = c(1 + \cos\theta).$ **18.** $\rho = e^{\sqrt{c - \theta^2}}.$ **19.** $\rho^m \cos m\theta = c.$
20. $2\rho = \log\tan\theta + c.$ **21.** $(\rho - 1/\rho)\sin\theta = c.$

SECTION III·4

4. $x = -\dfrac{gt}{k^2} + \dfrac{g}{k^4}(1 - e^{-k^2 t}).$

6. $v = v_0 e^{kt}$, $x = \dfrac{v_0}{k}(e^{kt} - 1).$

If $k > 0$, v and x grow indefinitely.
If $k < 0$, v diminishes continually, approaching 0 asymptotically,
x grows continually, approaching $-v_0/k$ asymptotically.

7. (a) $i = \dfrac{E}{R}(1 - e^{-Rt/L});$ (c) $i = \dfrac{e^{-Rt/L}}{L}\displaystyle\int_0^t E(t)e^{Rt/L}\,dt;$
(d) $i = i_0 e^{-R(t-t_0)/L}.$
8. 1.68 min.; 2.68 min.; 6 min.
9. $\log\dfrac{x}{x_0} = -\dfrac{\log 2}{1500}t;\ \dfrac{x}{x_0} = 2^{-2/3} = 0.63.$
11. $x = a^2kt/(akt + 1).$
12. $x = a - a/(2a^2kt + 1)^{1/2}.$
14. $x = a^2k\mu t/(akm\mu t + 1)$, where $\mu = b/a = n/m.$
17. (b) $(x + a)/r_1 - (x - a)/r_2 = c$, or $\cos\theta_1 - \cos\theta_2 = c.$
19. $y = -\dfrac{P}{6EI}(3lx^2 - x^3);\ y_0 = -\dfrac{Pl^3}{3EI},\ y_0' = -\dfrac{Pl^2}{2EI}.$
20. $y = -\dfrac{w}{24EI}(x^4 - 4l^3x + 3l^4);\ y_0 = -\dfrac{wl^4}{8EI},\ y_0' = \dfrac{wl^3}{6EI}.$
21. $y = -\dfrac{w}{24EI}(x^4 - 4lx^3 + 6l^2x^2);\ y_0 = -\dfrac{wl^4}{8EI},\ y_0' = -\dfrac{wl^3}{6EI}.$
22. $y = -\dfrac{P}{48EI}(3l^2x - 4x^3);\ y_0 = -\dfrac{Pl^3}{48EI},\ y_0' = -\dfrac{Pl^2}{16EI}.$
23. $y = -\dfrac{w}{24EI}(x^4 - 2lx^3 + l^3x);\ y_0 = -\dfrac{5wl^4}{384EI},\ y_0' = -\dfrac{wl^3}{24EI}.$

SECTION IV·1

4. $y = cx,\ x^2 + y^2 = c.$ **5.** $y^2 = 2x(x + c)^2.$ **6.** $2x = ce^y - \dfrac{1}{ce^y}.$
7. $xy - cy + 1 = 0,\ x + y + 1 = ce^x,\ x - y - 1 = ce^{-x}.$
8. $x^2 + y^2 + 4cx + 2c^2 = 0.$
9. $(y - x)^3(y + x) = c,\ (y + x)^3(y - x) = c.$

SECTION IV·2

3. $x^2 - 2cy - c^2 = 0.$
4. $(x + c)^3 = (y - c)^2.$
5. $xy = c^2x + c.$
6. $4(x^2 + y)^3 = (2x^3 + 3xy + c)^2.$
7. $2y - cx^2 + 3c^3 = 0.$
8. $x^2 = 2cy + a^2 + c^2.$
9. $cxy + ax + c^2 = 0.$
10. $y - cx^2 + ac^2 = 0.$

SECTION IV·3

1. $y^2 - 2cx + a^2c^2 = 0.$

2. $x = \dfrac{-cy'(2y'^2 + 3)}{(1 + y'^2)^{3/2}},$

$y = \dfrac{c}{(1 + y'^2)^{3/2}}$; or

$(x^2 + y^2 - 4c_1)^3 + 27c_1y^4 = 0.$

3. $y^3 = 3cx - c^2.$
4. $y = c(x - c)^2.$
5. $xy - cy + c^2 = 0.$
6. $4y^3 = c(3x - 2c)^2.$
7. $y^2 = 2cx + c^3.$
8. The circles $(x - c)^2 + y^2 = c$

SECTION IV·4

2. $c^2x - cy + 3 = 0.$
3. $c^2(x^2 + 1) - 2cxy + y^2 = 1.$
4. $c^2x^2 - 2c(xy - 4) + y^2 = 0.$
9. $cy^2 - c^2x^2 = a^2.$
10. $x^2 + y^2 = c(x + y) - c^2/4,$
or $(x - c_1)^2 + (y - c_1)^2 = c_1^2.$
11. $y = cx^2 + c^2x.$
15. $e^{2y} = ce^x + c^2.$
16. $e^y = c(e^x + 1) + c^3.$
19. $y^2 - cx^2 + c^2x = 0.$
20. $(x - c)^2 + y^2 = 1 - c^2.$

SECTION IV·5

1. $(x - c)^2 + y^2 = a^2.$
2. $cy = c^2(x - b) + a.$
3. $cxy + x + c^2 = 0.$
4. $x^2 + c(x - 3y) + c^2 = 0.$
5. $(x - y + 1)^2 + 2c(x + y + 1) + c^2 = 0.$
6. $y^2 = cx^2 - a^2c/(c + 1)$, or $\dfrac{x^2}{c_1} + \dfrac{y^2}{c_1 - a^2} = 1.$
7. $y(1 - \cos x) = c,\ y(1 + \cos x) = c.$
8. $(y - cx)^2 = 1 - c^2.$
9. $y - cx - 2c = 0,\ y - cx + 2c = 0.$
10. $y^2 = cx^2 + c^2.$
11. $c^2x^2 - cy + x = 0.$
12. $xy - cx + c^2 = 0.$
13. $x^2 - y^2 = 2cx + c^2.$
14. $2y = x[e^{x+c} - e^{-(x+c)}].$
15. $\theta = \displaystyle\int \frac{f(\rho)\,d\rho}{\rho\sqrt{\rho^2 - [f(\rho)]^2}} + c.$
16. $\rho = ce^{\theta\sqrt{1-k^2}/k}.$

17. $x = 2k\left(\tan^{-1} y' + \dfrac{y'}{1 + y'^2}\right) + c,\quad y = \dfrac{2ky'^2}{1 + y'^2}$; or, if we put $y' = \tan(t/2)$, $x = k(t + \sin t) + c$, $y = k(1 - \cos t)$, a family of cycloids

generated by a circle of radius k. If $x = 0$ when $y' = 0$, then $c = 0$. This makes the fixed point the vertex of the arch of the cycloid. The line, along the lower side of which the circle rolls, is $y = 2k$.

18. The same family of cycloids as in Ex. 17.

SECTION V·3

1. $\dfrac{x^2}{a^2 + k} + \dfrac{y^2}{k} = 1$. According as k is positive or negative, the curve is an ellipse or an hyperbola, having the fixed points as foci, and $\sqrt{\pm k}$ as the semi-conjugate axis.

2. $x^2 + y^2 = k^2$, a circle.

3. $2xy = a^2$, an equilateral hyperbola.

4. $(x - y)^2 - 2a(x + y) + a^2 = 0$, a parabola.

5. (*a*) $(y - cx)^2 + 4c = 0$, g.s.*, $xy - 1 = 0$, s.s.
(*b*) $(x + y - c)^2 = 4xy$, g.s., $xy = 0$, s.s.
(*c*) $c^2x - cy + 1 = 0$, g.s., $y^2 = 4x$, s.s.
(*d*) $(y - cx)^2 = b^2 + a^2c^2$, g.s., $b^2x^2 + a^2y^2 = a^2b^2$, s.s.
(*e*) $y^2 + 2cx + c^2 = 0$, g.s., $x^2 - y^2 = 0$, s.s.
(*f*) $x^2 + y^2 - 2cxy + c^2 = 1$, g.s., $(x^2 - 1)(y^2 - 1) = 0$, s.s.

SECTION V·4

4. $4y^3 = 9(x + c)^2$, g.s., $y = 0$, c.l.

5. $y^3 - y^4 = (x + c)^2$, g.s., $y = 1$, s.s., $y = 0$, c.l., $y = \frac{3}{4}$, t.l.

6. $y + c(x - y) + c^2(a - x) = 0$, g.s., $(x + y)^2 = 4ay$, s.s.

7. §IV·1, **3.** $y^2 - 1 = 0$, s.s.
8. $x^2 - y^2 = 0$, s.s., $y = 0$, t.l.
§IV·2, **1.** $x(x + 4y) = 0$, s.s.
4. $27(x + y) = 4$, s.s., $x + y = 0$, c.l.
5. $4x^2y + 1 = 0$, s.s., $x = 0$, t.l.
6. $x^2 + y = 0$, c.l.
7. $x^6 - 81y^2 = 0$, s.s., $x = 0$, t.l.
8. $x^2 + y^2 = a^2$, s.s., $x = 0$, t.l.
10. $x^4 - 4ay = 0$, s.s., $x = 0$, t.l.
§IV·3, **1.** $x^2 - a^2y^2 = 0$, s.s.
3. $9x^2 - 4y^3 = 0$, s.s.
4. $4x^3 - 27y = 0$, s.s., $y = 0$, s.s. and t.l.

---

* In the answers to the exercises of this and the next section, the following abbreviations are used: g.s. for general solution, s.s. for singular solution, p.c. for particular curve, t.l. for tac-locus, n.l. for nodal locus, c.l. for cuspidal locus.

5. $y - 4x = 0$, s.s., $y = 0$, p.c.
6. $x^3 - 2y^3 = 0$, s.s., $y = 0$, c.l.
7. $32x^3 + 27y^4 = 0$, s.s., $y = 0$, t.l.
8. $4y^2 = 4x + 1$, s.s., $y = 0$, t.l.
§IV·4, **1.** $x^2 + y^2 = 1$, s.s.
2. $y^2 - 12x = 0$, s.s.
3. $y^2 - x^2 = 1$, s.s.
4. $xy = 2$, s.s.
9. $y^4 - 4a^2x^2 = 0$, s.s., $y = 0$, t.l.
10. $xy = 0$, s.s., $x - y = 0$, t.l.
11. $x^3 + 4y = 0$, s.s., $x = 0$, t.l. and p.c.
19. $x^3 - 4y^2 = 0$, s.s., $y = 0$, t.l., $x = 0$, p.c. and t.l.
20. $x^2 + 2y^2 - 2 = 0$, s.s., $y = 0$, t.l.
§IV·5, **2.** $y^2 = 4a(x - b)$, s.s.
3. $xy^2 - 4 = 0$, s.s., $x = 0$, p.c.
4. $(x - y)(x + 3y) = 0$, s.s.
5. $(x + 1)y = 0$, s.s.
8. $y^2 - x^2 = 1$, s.s.
12. $x - 4y = 0$, s.s., $x = 0$, p.c.
13. $2x^2 - y^2 = 0$, s.s., $y = 0$, t.l.

SECTION VI·2

3. $y = c_1 + c_2e^{x} + c_3e^{-x}$.
4. $y = c_1e^{x} + c_2e^{-x} + c_3e^{2x}$.
5. $y = c_1e^{2ix} + c_2e^{-2ix}$.
6. $y = c_1e^{-x} + c_2e^{2x} + c_3e^{-2x}$.
7. $y = c_1e^{x} + c_2e^{-x} + c_3e^{ix} + c_4e^{-ix}$.
8. $y = c_1e^{x/3} + c_2e^{-x/2}$.

SECTION VI·3

2. $y = (c_1 + c_2x)e^{x} + c_3e^{-x}$.
3. $y = c_1e^{x} + (c_2 + c_3x + c_4x^2)e^{-x}$.
4. $y = c_1 + (c_2 + c_3x)e^{3x}$.
5. $y = c_1 + (c_2 + c_3x + c_4x^2)e^{-x}$.
6. $y = c_1 + c_2x + c_3e^{x} + c_4e^{-x}$.

SECTION VI·4

2. $y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x$.
3. $y = c_1e^{x} + c_2e^{-x} + c_3 \cos x + c_4 \sin x$.
4. $y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x$.
5. $y = c_1 + e^{x/2}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right)$.

SECTION VI·7

2. $y = c_1e^{-x} + c_2e^{-2x} + e^{-2x}e^{e^x}$.
3. $y = (c_1 + c_2x + c_3x^2)e^{-x} + e^{-x}(20x^3 - x^5)/60$.
4. $y = c_1e^{-x} + c_2e^{2x} + (\cos x - 3 \sin x)/10$.
5. $y = (c_1 + c_2x)e^x - e^x \log (1 - x)$.

SECTION VI·8

1. $y = c_1e^x + c_2e^{2x} - xe^x$.
2. $y = c_1e^x + c_2e^{-x} + c_3e^{3x} + (9x^2 + 6x + 20)/27$.
3. $y = (c_1 + c_2x)e^x + c_3e^{2x} - (2x + 5)/4$.

SECTION VI·9

2. $y = c_1 \cos x + c_2 \sin x - \cos x \log (\sec x + \tan x)$.
3. $y = c_1 \cos x + c_2 \sin x + \cos x \log (\csc x + \cot x) - 2$.
4. $y = c_1e^x + c_2e^{-x} + (e^x - e^{-x}) \log (e^x - 1) - xe^x + 1$.
5. $y = c_1 \cos x + c_2 \sin x + c_3 + \log \sec x - \sin x \log (\sec x + \tan x)$.

SECTION VI·10

3. $y = c_1 \cos x + c_2 \sin x + e^x + x^3 - 7x$.
4. $y = (c_1 + c_2x)e^{-x} + \frac{1}{3}e^{2x} - \frac{1}{2} \sin x$.
5. $y = c_1e^x + c_2e^{-x/2} \cos \left(\frac{x}{2}\sqrt{3} + c_3\right) - x^2$.
6. $y = c_1 + c_2e^{-x} + c_3e^{3x} - x^3/3 + 2x^2/3 - 14x/9 + (\sin x + 2 \cos x)/10$.
7. $y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x} + x^2e^x/8 + 4$.
8. $y = c_1 + c_2e^{2x} + xe^{2x}/2 - e^x - x^2/4 - 7x/4$.
9. $y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x - \frac{1}{8}x^2 \cos x$.

SECTION VI·11

2. $y = (c_1 \cos \log x + c_2 \sin \log x)x + c_3x^{-1} + 5x + 2x^{-1} \log x$.
3. $y = (c_1 + c_2 \log x)x^{-1} - x^{-1} \log (1 - x)$.
4. $y = c_1(x + 1)^2 + c_2(x + 1)^3 + (3x + 2)/6$.
5. $y = (x + 2)[c_1 \cos \log (x + 2) + c_2 \sin \log (x + 2)] + c_3 + x$.

SECTION VI·12

1. $y = c_1e^{2x} + c_2e^{3x} + (\cos x - \sin x)/10 + xe^{2x}$.
2. $y = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}e^x \cos x$.

3. $y = (c_1 + c_2x)e^{-x} + 2x^3 - 12x^2 + 36x - 48 - xe^{3x}/16 + e^{3x}/32$
$- e^{-x} \sin x.$

4. $y = (c_1 + c_2x + c_3x^2)e^{-x} + x^4e^{-x}/24 - e^x/8.$

5. $y = c_1 + c_2e^{2x} + c_3e^{-2x} - x^3/12 - x/8 - 3xe^{2x}/8.$

6. $y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x} + \frac{1}{4}\cos x.$

7. $y = (c_1 + c_2 \log x) \sin \log x + (c_3 + c_4 \log x) \cos \log x + (\log x)^2$
$+ 2 \log x - 3.$

8. $y = c_1 + (c_2 + c_3x)e^{-x} + x^3/3 - 5x^2/2 + 8x + \log x.$

9. $y = c_1 \cos 2x + c_2 \sin 2x + (1 - x \sin 2x)/8.$

10. $y = c_1 \cos x + c_2 \sin x + \sin x \log (\tan x + \sec x) - 1.$

11. $y = (c_1 + c_2x + c_3x^2)e^x - x - 3 - x^6e^x/120.$

12. $y = (c_1 + c_2x + c_3x^2)e^x + c_4e^{-2x} + e^{3x}/40.$

13. $y = c_1 \cos x + c_2 \sin x + (x^2 \sin x + x \cos x)/4.$

14. $y = (c_1 + c_2 \log x)x + c_3x^{-1} + \dfrac{\log x}{4x}.$

15. $y = c_1e^x + c_2e^{-x/2} \cos \left(\dfrac{x\sqrt{3}}{2} + c_3\right) + xe^x(x - 2)/6$
$- (\cos 2x + 8 \sin 2x)/130 - 1/2.$

16. $y = (c_1 + c_2x)e^x - \frac{1}{2} \sin x + e^x(6x^2 + x^4)/12.$

SECTION VI·13

3. $\theta = \sqrt{\theta_0{}^2 + \omega_0{}^2/\mu^2} \cos [\mu t - \tan^{-1} (\omega_0/\mu\theta_0)].$

4. $\theta = \theta_0 \cos \mu t$; $\theta = \dfrac{\omega_0}{\mu} \sin \mu t$; $\theta = \theta_0 \cos \mu t + \dfrac{\omega_0}{\mu} \sin \mu t.$

5. For (8), $\theta = \theta_0e^{-kt}\left(\cos \lambda t + \dfrac{k}{\lambda} \sin \lambda t\right)$; $\theta = \dfrac{\omega_0}{\lambda} e^{-kt} \sin \lambda t$;

$$\theta = e^{-kt}\left(\theta_0 \cos \lambda t + \frac{k\theta_0 + \omega_0}{\lambda} \sin \lambda t\right).$$

For (9), $\theta = \dfrac{\mu\theta_0}{\lambda} e^{-kt} \cos \left(\lambda t - \tan^{-1} \dfrac{k}{\lambda}\right)$, $\mu = \sqrt{k^2 + \lambda^2}$;

$$\theta = \frac{\omega_0}{\lambda} e^{-kt} \sin \lambda t;$$

$$\theta = \frac{\sqrt{\mu^2\theta_0{}^2 + 2k\theta_0\omega_0 + \omega_0{}^2}}{\lambda} e^{-kt} \cos \left(\lambda t - \tan^{-1} \frac{k\theta_0 + \omega_0}{\lambda\theta_0}\right).$$

6. $\theta = \theta_0(1 + kt)e^{-kt}$; $\theta = \omega_0te^{-kt}.$

7. $\theta = [\theta_0 + (\omega_0 + k\theta_0)t]e^{-kt}$, $\theta' = [\omega_0 - k(\omega_0 + k\theta_0)t]e^{-kt}.$

If θ_0 and $\omega_0 + k\theta_0$ have opposite signs, θ vanishes when $t = -\theta_0/(\omega_0+k\theta_0)$. It then changes sign and increases in absolute value until $t = \omega_0/k(\omega_0+k\theta_0)$, when $d\theta/dt$ vanishes. From then on, θ diminishes continually in absolute value, approaching zero asymptotically.

If θ_0 and $\omega_0 + k\theta_0$ have the same sign, θ does not vanish at a finite time.

When θ_0 and ω_0 have the same sign, θ increases in absolute value until $t = \omega_0/k(\omega_0 + k\theta_0)$, after which it diminishes continually in absolute value, approaching zero asymptotically. If θ_0 and ω_0 have opposite signs, θ diminishes continually in absolute value, approaching zero asymptotically.

8. $\theta = \dfrac{e^{-kt}}{2\nu}[\{(\nu + k)\theta_0 + \omega_0\}e^{\nu t} + \{(\nu - k)\theta_0 - \omega_0\}e^{-\nu t}].$

9. $V = \dfrac{C}{\lambda^2 - p^2} e^{-kt} \cos pt,$ if $p \neq \lambda;$

$V = \dfrac{C}{2p} te^{-kt} \sin pt,$ if $p = \lambda.$

In the latter case, the maximum value of the amplitude equals $C/2pke$, and is attained when $t = 1/k$. The amplitude, from then on, approaches zero asymptotically.

10. (*a*) $A = \theta_0,\ B = (\theta_1 - \theta_0 \cosh \mu L)/\sinh \mu L.$

$$\therefore \theta = \frac{\theta_1 \sinh \mu x + \theta_0 \sinh \mu(L - x)}{\sinh \mu L};$$

(*b*) $c_1 = 0,\ c_2 = \theta_0. \quad \therefore \theta = \theta_0 e^{-\mu x}.$

11. $c_1 = \dfrac{r_0 r_1(\theta_0 - \theta_1)}{r_1 - r_0},\ c_2 = \dfrac{r_1\theta_1 - r_0\theta_0}{r_1 - r_0}.$

If $\theta_1 = 0,\ \theta = \dfrac{\theta_0 r_0 r_1}{r_1 - r_0}\left(\dfrac{1}{r} - \dfrac{1}{r_1}\right).$

Amount of flow per unit area $\equiv W = -k\, d\theta/dr.$

Quantity flowing across surface of radius $r = 4\pi r^2 W = \dfrac{4\pi k\theta_0 r_0 r_1}{r_1 - r_0}.$

12. $c_1 = \dfrac{\theta_0 \log r_1}{\log r_1 - \log r_0},\ c_2 = \dfrac{-\theta_0}{\log r_1 - \log r_0}.$

$\theta = \theta_0 \dfrac{\log r_1 - \log r}{\log r_1 - \log r_0}.$

Quantity $= 2\pi r W = \dfrac{2\pi k\theta_0}{\log r_1 - \log r_0}.$

14. If $1 - k/h^2 = \mu^2, \qquad u = c_1 \cos \mu t + c_2 \sin \mu t;$

if $1 - k/h^2 = -\nu^2, \quad u = c_1 e^{\nu t} + c_2 e^{-\nu t}.$

15. The equation of motion along the rod is

$$r = r_0 \cosh \omega t + \left(\frac{v_0}{\omega} - \frac{g}{2\omega^2}\right) \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t.$$

If $r_0 = 0$ and $v_0 = g/2\omega$, we have simple harmonic motion.

The equation of motion in space is obtained in polar coördinates by replacing r by ρ and ωt by θ. In general, this is the equation of a spiral. In the special case, we have the circle $\rho = \dfrac{g}{2\omega^2} \sin \theta.$

16. $s - l = (s_0 - l) \cos (\sqrt{g/e}\, t).$

17. $y = e \sec (\mu l/2) \cos [\mu(x - l/2)], \quad \mu = \sqrt{P/EI}.$
Greatest value of $y = e \sec (\mu l/2)$; that of $y' = e\mu \tan (\mu l/2).$

SECTION VII·1

2. $y = c_1e^x + c_2x^2e^x + x.$

3. $y = c_1x + c_2(1 + x \tan^{-1} x) + x \log \sqrt{1 + x^2}.$

4. $y = c_1x + c_2e^x + x^2 + 1.$

5. $y = c_1e^x + c_2e^x \log x + x + 3.$

6. $y = c_1e^{-x} + c_2x^2e^{-x} + x^2 - 2.$

7. $y = c_1x^3 + c_2xe^x + x.$

9. $y = \sec x\, (c_1e^{ax} + c_2e^{-ax}) - \dfrac{\sin x}{a^2 + 4}.$

10. $y = (c_1x^2 + c_2x^{-1} - x)e^{x^2}.$

11. $xy = c_1e^x + c_2e^{-x} + xe^x.$

12. $y = e^x(c_1x^3 + c_2x^{-2}) + x^3.$

SECTION VII·2

2. $y = c_1 \sin (\sin x) + c_2 \cos (\sin x).$

3. $y = c_1x\sqrt{1 - x^2} + c_2(1 - 2x^2) + x + 1 - 2x\sqrt{1 - x^2} \sin^{-1} x.$

4. $y = c_1 \cos \dfrac{1}{2x^2} + c_2 \sin \dfrac{1}{2x^2} + \dfrac{1}{x^2}.$

5. $y = c_1e^{2x^2} + c_2e^{-x^2} - \frac{1}{3}x^2e^{-x^2}.$

SECTION VII·3

1. $y = c_1e^x + c_2(x^3 + 3x^2 + 6x + 6) + x.$

2. $y = c_1e^x + c_2e^{3x}(4x^3 - 42x^2 + 150x - 183).$

3. $y = (c_1e^x + c_2e^{-x})/x^2.$
4. $y = c_1x + c_2(x^2 - 1) + 2.$

5. $y = c_1e^{x^2} + c_2e^{-x^2} - \log x.$
6. $y = (c_1 \cos x + c_2 \sin x)x^2 + x^3.$

7. $y = c_1e^{2x^2} + c_2e^{-x^2/2} - x^2 + 3/2.$
8. $2y = x(c_1e^{2x} + c_2 - x).$

9. $y = c_1x^n \cos (ax + c_2).$
10. $y = c_1 \cos (n/x) + c_2 \sin (n/x).$

11.
$$y = c_1x + c_2\left(1 - \frac{1}{2\cdot 3x^2} + \frac{1}{2!2^2\cdot 5x^4} - \cdots + \frac{(-1)^n}{n!2^n(2n + 1)x^{2n}} + \cdots\right).$$

12. $y = c_1e^{-x} + c_2x^2 - x^2e^{-x}.$

SECTION VIII·2

3. $y = c_1 + c_2e^{-x^2/2} + x.$
4. $y = c_1 + c_2x + (x - 2)e^x.$

5. $(x - c_1)^2 + (y - c_2)^2 = a^2.$
6. $y = \log \cos (x + c_1) + c_2.$

7. $x^2 - 2c_1x + (y - c_2)^2 = 0.$

SECTION VIII·3

2. $(x - c_1)^2 + y^2 = c_2.$

3. $e^y = 4ac^2e^{cx}/(1 - ae^{cx})^2,$
or $e^y = c^2 \sec^2[(cx + a)/2]$
$= 2c^2/[1 + \cos(cx + a)].$

4. $ay + 2 = be^{ax}.$

5. $(y - c)^2 = ax + b$
and $y = ax + b.$

SECTION VIII·4

1. $\log y = x^3 + c_1x + c_2.$

2. $x^2y^{4a} = be^{a^2x^2}.$

3. $y^2 = ax^2 + b.$

4. $\log y = c_1 + c_2/x.$

SECTION VIII·5

1. $(\log x + c_1)^2 + (y/x)^2 = c_2.$

2. $y/x = \log\cos(c_1 - \log x) + c_2.$

3. $(\log x + c_1)^2 - y^2/x^4 = c_2.$

4. $x^2y^2 + \log x = ax^2 + b.$

SECTION VIII·6

1. $3xy + y^3 = c_1x^2 + c_2x + c_3.$

2. $(x - 1)y = x^2 - 2x + c_1 + c_2e^{-x}.$

3. $xy = x^2 - 2x + c_1 \log(x + 1) + c_2.$

4. $xy\sqrt{x^2 - 1} = c_1\sqrt{x^2 - 1} + c_2 \log(x + \sqrt{x^2 - 1}) + c_3.$

5. $y^2 - 2xy = x^2 + 2\sin x + c_1x + c_2.$

6. $x^3y^2 = c_1x^2 + c_2x + c_3.$

7. $(1 - x)^2y = x^3 - x^2 + x/3 + c_1(4x - 3) + c_2x^4.$

8. $y = c_1 \cos(1/x) + c_2 \sin(1/x) + \log x + c_3.$

9. $x^3y = c_1(1 - 5x + 10x^2) + c_2(10x^3 - 5x^4 + x^5).$

SECTION VIII·7

2. $y = c_1x + c_2xe^{-1/2x^2}.$

3. $xy = c_1x + (c_2 - c_1 \sin^{-1} x)\sqrt{1 - x^2}.$

4. $y = c_1 \cos x^2 + c_2 \sin x^2 + c_3x^2.$

5. $\tan y = c_1 \cot x + c_2.$

SECTION VIII·8

3. $y = c_1x + (c_2x + c_3x^2)e^x.$

SECTION VIII·9

2. $y = x \log[c_1x/(c_2 + x)].$

3. $\log y = c_1e^x + c_2e^{-x} + x^2 + 2.$

4. $y = c_1 \cot x + c_2(1 - x \cot x).$

5. $y^{1/2} = c_1x + c_2x^2.$

6. $y^2 = c_1 + c_2x^{-2} + x^4/6.$

SECTION VIII·10

1. $y = c_1 - \log\cos(x + c_2)$.

2. $y = c_1 + c_2 \sin^{-1} x + (\sin^{-1} x)^2$.

3. $(a + y)/(a - y) = e^{a(x+b)}$ and $y = c$.

4. $(1 + x^3)y = c_1x^2 + c_2x + c_3$.

5. $(x - 1)^5y = c_1(x^4 - 6x^2 + 2x - 1/3 - 4x^3 \log x) + c_2x^3$.

6. $\log y = 1 + 1/(c_1x + c_2)$

7. $e^y = c \sec[c(x + a)]$,
or $e^y = 2c/[e^{c(x+a)} - e^{-c(x+a)}] = c \operatorname{csch}[c(x + a)]$.

8. $5 \log y = c_1 + c_2x^5 + 8 \log x$. **9.** $y = x^2 + c_1e^{x^2} + c_2$.

10. $(x - c_1)^2 + (y - c_2)^2 = c_3$ and $y = ax + b$.

11. $(x - c_1)^2 + c_2(y - c_3)^2 = c_4$ and $y = ax^2 + bx + c$.

12. $\log y = c_1 + c_2x^2 + x + x^2 \log x$.

13. $x^2y + xy^2 = c_1x + c_2 - x^4/12$.

14. $y = c_1 + c_2\sqrt{x^2 - 1} + x^2/2$. **15.** $y = c_1 + c_2e^{1/x} + 1/x$.

16. $y = c_1 \sin^2 x + c_2 \cos x + c_2 \sin^2 x \log(\csc x + \cot x)$.

17. (*a*) The circles $(x - c_1)^2 + y^2 = c_2$,
(*b*) the catenaries $2y = a[e^{(x-b)/a} + e^{-(x-b)/a}]$.

18. (*a*) The cycloids $x - a = b \cos^{-1}[(b - y)/b] - \sqrt{2by - y^2}$,
(*b*) the parabolas $(x - a)^2 = 2by - b^2$.

19. The central conics $kcy^2 - c^2(x - a)^2 = k$.

20. $2y = \dfrac{na^{1/n}(a - x)^{(n-1)/n}}{1 - n} + \dfrac{na^{-1/n}(a - x)^{(n+1)/n}}{1 + n} - \dfrac{2na}{1 - n^2}$, if $n \neq 1$;
$\pm 4ay = (a - x)^2 - 2a^2 \log[(a - x)/a] - a^2$, if $n = 1$.

21. $v \equiv l\, d\theta/dt = \sqrt{2gl(\cos\theta - \cos\alpha)}$.

22. (*a*) $v = -k\sqrt{2(a - x)/ax}$.
(*b*) $t = (\sqrt{a/2}/k)(a \cos^{-1}\sqrt{x/a} + \sqrt{ax - x^2})$.
(*c*) $a/2$.
(*d*) $v = -\sqrt{2gRh/(R + h)}$; if $h = \infty$, $v = -\sqrt{2gR} = -6.96$ mi./sec.
[Here $R = 4000$ miles, $g = 32$ ft./sec.^{2}]

SECTION IX·3

3.
$$y = c_0\left(1 + \frac{x^3}{3!} + \frac{4x^6}{6!} + \frac{4\cdot 7x^9}{9!} + \cdots + \frac{4\cdot 7\cdots(3r - 2)x^{3r}}{(3r)!} + \cdots\right)$$
$$+ c_1\left(x + \frac{2x^4}{4!} + \frac{2\cdot 5x^7}{7!} + \cdots + \frac{2\cdot 5\cdots(3r - 1)x^{3r+1}}{(3r + 1)!} + \cdots\right).$$

4. $y = c_0 + c_1x + c_0^2x^2/2 + c_0(2c_1 + 1)x^3/3! + 2(c_0^3 + c_1 + c_1^2)x^4/4!$
$+ 5c_0^2(2c_1 + 1)x^5/5! + \cdots$.

6. $y = c_0(1 - 2x^2 + x^4)$
$+ c_1[x - x^3 + x^5/5 + x^7/5\cdot 7 + 3x^9/5\cdot 7\cdot 9 + \cdots$
$+ 3x^{2n+1}/(2n - 3)(2n - 1)(2n + 1) + \cdots]$.

SECTION IX·5

2. $$\text{P.I.} = \frac{x^{s+1}}{(s+1)(s-3)}\left[1 + \frac{s-1}{s-2}x + \frac{s}{s-2}x^2 + \cdots + \frac{s+r-2}{s-2}x^r + \cdots\right].$$

If $s = -1, 2, 3$, no particular integral of this form exists.

3. $$y = A\left[1 - \frac{x^3}{3!} + \frac{4^2x^6}{6!} - \frac{4^2 \cdot 7^2 x^9}{9!} + \cdots + (-1)^r \frac{4^2 \cdot 7^2 \cdots (3r-2)^2 x^{3r}}{(3r)!} + \cdots\right]$$
$$+ B\left[x - \frac{2^2x^4}{4!} + \frac{2^2 \cdot 5^2 x^7}{7!} - \cdots + (-1)^r \frac{2^2 \cdot 5^2 \cdots (3r-1)^2 x^{3r+1}}{(3r+1)!} + \cdots\right]$$
$$+ \frac{x^{s+2}}{(s+1)(s+2)} - \frac{(s+3)x^{s+5}}{(s+1)(s+2)(s+4)(s+5)} + \frac{(s+3)(s+6)x^{s+8}}{(s+1)(s+2)(s+4)(s+5)(s+7)(s+8)} - \cdots.$$

4. $y = A(6 - 8x + 3x^2) + Bx^{-2}$.

5. $y_1 = 1 - x/2! + x^2/4! - \cdots = \cos\sqrt{x}$,
$y_2 = x^{1/2}[1 - x/3! + x^2/5! - \cdots = \sin\sqrt{x}$.

6. $$y_1 = 1 - 3x + 3x^2/1\cdot 3 + 3x^3/3\cdot 5 + 3x^4/5\cdot 7 + \cdots + 3x^r/(2r-3)(2r-1) + \cdots,$$
$y_2 = x^{1/2}(1 - x)$.

7. $$y_1 = 1 + 3x^2 + \frac{3x^4}{5} - \frac{3\cdot 1}{5\cdot 9}x^6 + \frac{3\cdot 1\cdot 3}{5\cdot 9\cdot 13}x^8 - \cdots + (-1)^r \frac{3\cdot 1\cdot 3\cdot 5\cdot 7\cdots(2r-5)}{5\cdot 9\cdot 13\cdots(4r-3)}x^{2r} + \cdots,$$
$$y_2 = x^{3/2}\left[1 + \frac{3x^2}{8} - \frac{3\cdot 1}{8^2\cdot 2!}x^4 + \frac{3\cdot 1\cdot 5}{8^3\cdot 3!}x^6 - \frac{3\cdot 1\cdot 5\cdot 9}{8^4\cdot 4!}x^8 + \cdots + (-1)^{r-1}\frac{3\cdot 1\cdot 5\cdot 9\cdots(4r-7)}{8^r\cdot r!}x^{2r} + \cdots\right].$$

8. $$y_1 = 1 + \frac{x}{3} + \frac{1\cdot 4}{3^2\cdot 2!}x^2 + \frac{1\cdot 4\cdot 7}{3^3\cdot 3!}x^3 + \cdots = (1 - x)^{-1/3},$$
$$y_2 = x^{7/3}\left[1 + \frac{8}{10}x + \frac{8\cdot 11}{10\cdot 13}x^2 + \cdots + \frac{8\cdot 11\cdot 14\cdots(3r+5)}{10\cdot 13\cdot 16\cdots(3r+7)}x^r + \cdots\right].$$

9. See Bessel's equation, §IX·10.

10. $$y_1 = 1 - \frac{x^3}{3!} - \frac{1\cdot4}{5\cdot3^2\cdot2!}x^6 - \cdots - \frac{1\cdot4\cdot7\cdots(3r-2)}{(3r-1)3^r r!}x^{3r} - \cdots,$$
$$y_2 = x.$$

11. $$y_1 = 1 - 3x^2,\ y_2 = x\left[1 - \frac{2}{1\cdot3}x^2 - \frac{3}{3\cdot5}x^4 - \frac{4}{5\cdot7}x^6 - \cdots - \frac{r+1}{(2r-1)(2r+1)}x^{2r} - \cdots\right].$$

12. $$\bar{y}_1 = 1 - \frac{x^{-2}}{3!} - \frac{x^{-4}}{5!} - \frac{3x^{-6}}{7!} - \frac{3\cdot5x^{-8}}{9!} - \cdots - \frac{3\cdot5\cdot7\cdots(2r-3)x^{-2r}}{(2r+1)!} - \cdots,$$
$$\bar{y}_2 = x - x^{-1}.$$

13. $$y_1 = x + \frac{x^3}{2\cdot5} + \frac{x^5}{2\cdot4\cdot5\cdot9} + \cdots + \frac{x^{2r+1}}{2^r r! 5\cdot9\cdot13\cdots(4r+1)} + \cdots,$$
$$y_2 = x^{1/2}\left[1 + \frac{x^2}{2\cdot3} + \frac{x^4}{2\cdot4\cdot3\cdot7} + \cdots + \frac{x^{2r}}{2^r r! 3\cdot7\cdot11\cdots(4r-1)} + \cdots\right],$$
$$\text{P.I.} = \frac{x^2}{3}\left[1 + \frac{x^2}{3\cdot7} + \frac{x^4}{3\cdot5\cdot7\cdot11} + \cdots + \frac{x^{2r}}{3\cdot5\cdot7\cdots(2r+1)\cdot7\cdot11\cdots(4r+3)} + \cdots\right].$$

14. $$y_1 = 1 + \frac{x}{3!} + \frac{3^2x^2}{5!} + \frac{3^2\cdot5^2x^3}{7!} + \cdots + \frac{3^2\cdot5^2\cdot7^2\cdots(2r-1)^2}{(2r+1)!}x^r + \cdots,$$
$$y_2 = x^{-1/2}.$$

SECTION IX·6

2. $$\text{P.I.} = a\left[\frac{x^{s+1}}{(s+1)^2} - \frac{x^{s+2}}{(s+1)^2(s+2)^2} + \frac{x^{s+3}}{(s+1)^2(s+2)^2(s+3)^2} - \cdots\right].$$

If s is a negative integer, no integral of this type exists.
If s is a positive integer, we can write it in the form

$$\text{P.I.} = a(s!)^2\left\{\frac{x^{s+1}}{[(s+1)!]^2} - \frac{x^{s+2}}{[(s+2)!]^2} + \cdots\right\}.$$

Comparing this with $a(s!)^2\, y_1$, we can replace P.I. with the polynomial

$$V = (-1)^s a(s!)^2\left[1 - x + \frac{x^2}{(2!)^2} - \cdots + (-1)^s\frac{x^s}{(s!)^2}\right];$$

i.e., the method of undetermined coefficients (§VI·10) applies.

3. $$y_1 = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots + (-1)^r\frac{x^{2r}}{2^{2r}(r!)^2} + \cdots,$$

$$y_2 = y_1 \log x + \frac{x^2}{2^2} - \frac{(1+\frac{1}{2})x^4}{2^4(2!)^2} + \frac{(1+\frac{1}{2}+\frac{1}{3})x^6}{2^6(3!)^2} - \cdots + \frac{(-1)^{r+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{r}\right)x^{2r}}{2^{2r}(r!)^2} + \cdots.$$

4. $y_1 = 1 + 2^2x + 3^2x^2 + \cdots + (r+1)^2x^r + \cdots,$
$y_2 = y_1 \log x - 2[1\cdot 2x + 2\cdot 3x^2 + 3\cdot 4x^3 + \cdots + r(r+1)x^r + \cdots].$

5. $y_1 = 1 - 2x + \dfrac{3x^2}{2!} - \dfrac{4x^3}{3!} + \cdots,$

$$y_2 = y_1 \log x + 2(2 - \tfrac{1}{2})x - 3(2 + \tfrac{1}{2} - \tfrac{1}{3})\frac{x^2}{2!} + 4(2 + \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4})\frac{x^3}{3!} - \cdots.$$

SECTION IX·7

2. $y_1 = -x^{-1}\left[\dfrac{x^2}{2} - \dfrac{x^4}{2^2\cdot 4} + \dfrac{x^6}{2^2\cdot 4^2\cdot 6} - \cdots + (-1)^{r+1}\dfrac{2rx^{2r}}{2^{2r}(r!)^2} + \cdots\right],$

$$y_2 = y_1 \log x + x^{-1}\left[1 + \frac{x^2}{2^2} - \left(\frac{2}{2} + \frac{1}{4}\right)\frac{x^4}{2^2\cdot 4} + \left(\frac{2}{2} + \frac{2}{4} + \frac{1}{6}\right)\frac{x^6}{2^2\cdot 4^2\cdot 6} - \cdots\right].$$

3. $y_1 = -\dfrac{x}{1} + \dfrac{x^2}{1^2\cdot 2} - \dfrac{x^3}{1^2\cdot 2^2\cdot 3} + \cdots + \dfrac{(-1)^r r x^r}{(r!)^2} + \cdots,$

$$y_2 = y_1 \log x + \left[1 + \frac{x}{1^2} - \left(\frac{2}{1^3\cdot 2} + \frac{1}{1^2\cdot 2^2}\right)x^2 + \left(\frac{2}{1^3\cdot 2^2\cdot 3} + \frac{2}{1^2\cdot 2^3\cdot 3} + \frac{1}{1^2\cdot 2^2\cdot 3^2}\right)x^3 - \left(\frac{2}{1^3\cdot 2^2\cdot 3^2\cdot 4} + \frac{2}{1^2\cdot 2^3\cdot 3^2\cdot 4} + \frac{2}{1^2\cdot 2^2\cdot 3^3\cdot 4} + \frac{1}{1^2\cdot 2^2\cdot 3^2\cdot 4^2}\right)x^4 + \cdots\right].$$

4. $y_1 = -x^{-2}\left[\dfrac{x^4}{2^2\cdot 4} - \dfrac{x^6}{2^3\cdot 4\cdot 6} + \dfrac{x^8}{2^3\cdot 4^2\cdot 6\cdot 8} - \dfrac{x^{10}}{2^3\cdot 4^2\cdot 6^2\cdot 8\cdot 10} + \cdots\right],$

$$y_2 = y_1 \log x + x^{-2}\left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{11x^6}{2^2\cdot 4^2\cdot 6^2} + \frac{31x^8}{2^2\cdot 4^2\cdot 6^2\cdot 8^2} - \cdots\right].$$

5. $y_1 = -x + \dfrac{x^2}{1!} - \dfrac{x^3}{2!} + \dfrac{x^4}{3!} - \cdots \equiv -xe^{-x},$

$$y_2 = y_1 \log x + x^{-1}\left[1 + x - x^2 + \frac{x^4}{2^2} - \left(\frac{1}{2^2\cdot 3} + \frac{1}{2\cdot 3^2}\right)x^5 + \left(\frac{1}{2^2\cdot 3\cdot 4} + \frac{1}{2\cdot 3^2\cdot 4} + \frac{1}{2\cdot 3\cdot 4^2}\right)x^6 - \cdots\right].$$

6. $y_1 = 1 + 2x + 3x^2 + 4x^3 + \cdots \equiv 1/(1-x)^2,$
$y_2 = 1 + 2x + 3x^2.$

SECTION IX·8

8. The answers, given for this set of exercises, are not unique.

$$F(1, 1, 1, x),\ 2F\left(-\frac{n}{2}, -\frac{n-1}{2}, \frac{1}{2}, x^2\right),$$

$$2nxF\left(-\frac{n-1}{2}, -\frac{n}{2}+1, \frac{3}{2}, x^2\right),$$

$$\text{Limit}_{\alpha,\,\beta\to\infty} F\left(\alpha, \beta, \tfrac{1}{2}, -\frac{x^2}{4\alpha\beta}\right),\ \text{Limit}_{\alpha,\,\beta\to\infty} 2F\left(\alpha, \beta, \tfrac{1}{2}, \frac{x^2}{4\alpha\beta}\right).$$

SECTION X·2

2. $z = ce^{x/y}$. **3.** $x^2 + y^2 + (z - c)^2 = a^2$. **4.** $x^2 + y^2 + z^2 = cx$.

5. $f(z) = -(1+z^2)+k(1+z^2)^{3/2}$, $\sin x - y(1-k\sqrt{1+z^2}) = c\sqrt{1+z^2}$.

SECTION X·3

1. $x^2 - (y - c)^2 = 2z$. **2.** $x + y + z = ce^{y-x}$. **3.** $x^2 - y - z = cx$.

4. $y^2 + z^2 = cx^2$. **5.** $\tan^{-1}(y/x) + \log z = c$. **6.** $\log x - x/yz = c$.

7. $x^2 - z \log y = c$.

SECTION X·4

3. $z(x + y) = c(x + z)$. **4.** $x^2 + y^2 = c(y + z)$.

5. $y + z = c(z - x)$. **6.** $x^2 + y^2 + z^2 = cy^4$.

7. $xy + yz + zx = c(x + y + z)$. **8.** $(y + z)/x + (z + x)/y = c$.

SECTION X·5

2. $(x^2 + 2yz)t^2e^{2t} = c$. **3.** $xy + yz + zt = c(x + t)$.

4. $x - y + z = c(y + z - t)$.

SECTION X·8

1. $yz + zx + xy = c$. **2.** $x^2 + y^2 = c(z + 1)^2$.

3. $e^x(x + y^2 + z^2) = c$. **4.** $x = z/(y + a) + c$.

5. $e^x(y + z) = c$. **6.** $(x^2 + y + z)e^{z^2} = c$.

7. $x^2 + xy^2 + x^2z - t = c$. **8.** $y/x + \log z = c$.

9. $(x-1)(y-1)(z-1)e^{x+y+z} = c$. **10.** $(x + 2y)(y - z)^2 = c$.

11. $(y + z)e^{x+y} = ct$. **12.** $xz + yt = c(y + z)$.

13. $xyz = ce^{x+y+z}$. **14.** $kx - hy = c(lx - hz)$.

15. $x^z = cy^t$. **16.** $y^4 + z^4 + t^4 + 4xyzt = cx$.

SECTION XI·2

1. $x = c_1e^{t/3} + c_2e^{-t/3} - 6t,\ y = -2c_1e^{t/3} - c_2e^{-t/3} + \frac{1}{2}e^t + 9t + 9.$

2. $x = (6c_2 - 2c_1 - 2c_2t)e^t - \frac{1}{3}c_3e^{-3t/2} - \frac{1}{3},$
$y = (c_1 + c_2t)e^t + c_3e^{-3t/2} - t/2.$

3. $x = (c_1 + c_2t)e^t + (c_3 + c_4t)e^{-t} + \cos t,$
$2y = (c_2 - c_1 - c_2t)e^t - (c_3 + c_4 + c_4t)e^{-t} - 2\cos t - \frac{1}{2}\sin t.$

4. $$x = \frac{c_2 - c_1}{2}\cos t - \frac{c_1 + c_2}{2}\sin t + c_3e^{-t/3} + \frac{2\omega\cos\omega t}{(9\omega^2 + 1)(\omega^2 - 1)} - \frac{(3\omega^2 + 1)\sin\omega t}{(9\omega^2 + 1)(\omega^2 - 1)},$$
$$y = c_1\cos t + c_2\sin t + c_3e^{-t/3} + \frac{2\omega\cos\omega t}{(9\omega^2 + 1)(\omega^2 - 1)} + \frac{6\omega^2\sin\omega t}{(9\omega^2 + 1)(\omega^2 - 1)}.$$

SECTION XI·3

6. $(x - y)t = c_1,\ x + y + 2 = c_2t.$

7. $x^2 + y^2 = c_1,\ \tan^{-1}\frac{y}{x} - \tan^{-1}t = \tan^{-1}c_2$ or $\frac{y - xt}{x + yt} = c_2.$

8. $x^2 - y^2 = c_1,\ 2(x + y) - t^2 = c_2.$

9. $y = c_1t,\ x^2 + y^2 + t^2 = c_2t.$

10. $x - y = c_1(x - t),\ (x + y + t)(x - t)^2 = c_2.$

11. $x - y - t = c_1,\ x^2 + y^2 = c_2t^2.$

12. $x^3y^3t = c_1,\ (x^3 + y^3)^3t^2 = c_2.$

13. $x^2 + y^2 + t^2 = c_1,\ xyt = c_2.$

14. $x^2 + y^2 + t^2 = c_1,\ x^3 - yt = c_2.$

15. $x^2 + y^2 + t^2 = c_1,\ y^2 - 2yt - t^2 = c_2.$

16. $xy = c_1,\ x^4 - 2xyt^2 - t^4 = c_2.$

SECTION XI·4

1. $y/x = c_1,\ z/x = c_2$; or $x/c_1 = y/c_2 = z/c_3$, where only the ratios of the c's play a rôle.

2. $y = c_1z,\ x^2 + y^2 + z^2 = c_2z.$

3. $\frac{x^{1/h}}{c_1} = \frac{y^{1/k}}{c_2} = \frac{z^{1/l}}{c_3}$, where only the ratios of the c's play a rôle.

4. $x^2 + z^2 = c_1,\ y^2 + z^2 = c_2.$

5. $y = c_1x,\ x^2 + y^2 + 2z^2 = c_2.$

6. $y = c_1x,\ x^2 + y^2 + z^2 = c_2.$

7. $y - x = c_1(z - x),\ (x + y + z)(y - x)^2 = c_2.$

SECTION XI·7

1. The path is a parabola lying in the vertical plane determined by the direction of the initial velocity. If the initial position of the particle is taken as the origin, and the horizontal and vertical lines of the plane through the origin are taken as the axes of x and y respectively, the equations of the path are

$$x = v_0t \cos \alpha,\ y = v_0t \sin \alpha - gt^2/2.$$

Or, eliminating t, we have

$$y = x \tan \alpha - gx^2/2v_0^2 \cos^2 \alpha.$$

2. $x = v_0 \cos \alpha(1 - e^{-kt})/k,\ y = (kv_0 \sin \alpha + g)(1 - e^{-kt})/k^2 - gt/k;$

3. $kx = ak \cos kt + v_0 \cos \alpha \sin kt,\ ky = v_0 \sin \alpha \sin kt.$

Eliminating t, we have

$$(x \sin \alpha - y \cos \alpha)^2 + a^2k^2y^2/v_0^2 = a^2 \sin^2 \alpha,$$

an ellipse with its center at the origin. Since x and y are periodic functions, of period $2\pi/k$, the motion is periodic, the period being independent of the dimensions of the ellipse.

4. $2kx = ka(e^{kt} + e^{-kt}) + v_0 \cos \alpha(e^{kt} - e^{-kt}),\ 2ky = v_0 \sin \alpha(e^{kt} - e^{-kt});$

or

$$kx = ka \cosh kt + v_0 \cos \alpha \sinh kt,\ ky = v_0 \sin \alpha \sinh kt.$$

Eliminating t, we have

$$(x \sin \alpha - y \cos \alpha)^2 - a^2k^2y^2/v_0^2 = a^2 \sin^2 \alpha,$$

an hyperbola with its center at the origin.

5. $p = a \cos (kt + b),\ q = a \sin (kt + b),\ r = c,\ k = c(C - A)/A.$

If $p = p_0, q = q_0, r = r_0$ when $t = 0$, then $a = \sqrt{p_0^2 + q_0^2}$,

$b = \tan^{-1}(q_0/p_0),\ c = r_0,\ k = r_0(C - A)/A.$

Angular velocity $\equiv \sqrt{p^2 + q^2 + r^2} = \sqrt{p_0^2 + q_0^2 + r_0^2}$, a constant.

6. If the respective factors of proportionality are a^2, b^2, c^2, the paths are given by two equations of the form

$$x^2/a^2 - y^2/b^2 = c_1 \quad \text{and} \quad x^2/a^2 - z^2/c^2 = c_2.$$

For the curve through the origin $c_1 = c_2 = 0$. Hence the path is one of the lines $x/a = \pm y/b = \pm z/c$.

If $x = x_0$, $y = y_0$, $z = z_0$ when $t = t_0$, the time is given by

$$t - t_0 = \frac{\pm 1}{bc} \int_{x_0}^{x} \frac{dx}{\sqrt{(x^2 - a^2c_1)(x^2 - a^2c_2)}}.$$

When $c_1 = c_2 = 0$,

$$t - t_0 = \frac{\pm 1}{bc}\left(\frac{1}{x_0} - \frac{1}{x}\right) = \frac{\pm 1}{ca}\left(\frac{1}{y_0} - \frac{1}{y}\right) = \frac{\pm 1}{ab}\left(\frac{1}{z_0} - \frac{1}{z}\right).$$

7. $x = \frac{mX}{eH^2}\left(1 - \cos\frac{eH}{m}t\right)$, $y = \frac{mX}{eH^2}\left(\frac{eH}{m}t - \sin\frac{eH}{m}t\right)$, a cycloid.

The initial position of the particle is taken as the origin.

SECTION XII·1

1. $z(p - q) + x - y = 0.$ **2.** $z^2q^2 - 2xzp - x^2 + z^2 = 0.$
3. $z = xp + yq + \sqrt{p^2 + q^2}.$ **4.** $pq = z.$
5. $(1 + q^2)r - (1 + p^2)t = 0$, or $pqr - (1 + p^2)s = 0.$
6. $r = 0$, or $t = 0$, or $z = xp + yq - xys.$ **7.** $zs + pq = 0.$

SECTION XII·2

2. $xzp + yzq = xy.$ **3.** $yp - xq = y^2 - x^2.$ **4.** $r - t = 0.$
5. $xr - (x + y)s + yt = \frac{x + y}{x - y}(p - q).$

SECTION XIII·1

3. $\phi(x^2 + y^2, xy + z) = 0.$ **4.** $\phi(x + y + z, x^2 + y^2 + z^2) = 0.$
5. $x + y + 2 = z\phi(xz - yz).$ **6.** $\phi(x^2 - y^2 - z^2, 2xy - z^2) = 0.$

SECTION XIII·2

2. $[y + z - \psi(x)]^2 = 4\psi(x)(x - y)$;
$x + z = 0$, $x - y = 0$ special solutions.
3. $x^2 + [z - y\psi(x)]^2 = 1$; $x \pm 1 = 0$ special solutions.
4. $z = y\sin^{-1}(x/y) + \psi(y)$; $y - x = 0$, $y + x = 0$ special solutions.
5. $z - y\psi(x) = [\psi(x) + x]^2$; $y^2 + 4xy + 4z = 0$ special solution.
7. $\phi(x^2 - 2y, y^2 - 4\sqrt{z}) = 0$; $z = 0$ special solution.
8. $\phi(\sqrt{x} - \sqrt{z}, \sqrt{y} - \sqrt{z}) = 0$; $x = 0$, $y = 0$, $z = 0$ special solutions.
9. $\phi(\tan^{-1}x - \tan^{-1}y, \tan^{-1}x - \sin^{-1}z) = 0$,

$$\text{or } \phi\left(\frac{x - y}{1 + xy}, \frac{x\sqrt{1 - z^2} - z}{\sqrt{1 - z^2} + xz}\right) = 0;$$

$z - 1 = 0$, $z + 1 = 0$ special solutions.
10. $\phi[2y - z, e^y \log(z - x - y)] = 0$; $z - x - y = 0$ special solution.

SECTION XIII·7

2. $y = (b - x)(a + yz)$.* **3.** $6z = (2x - a)^3 + 6a^2y + b$.
4. $(ax + y + b)^2 = (1 + a^2)(R^2 - z^2)$.

SECTION XIII·8

2. $\log z = ax + y/a + b$. **3.** $z = a \log x + \sqrt{a^2 - 1}\, y + b$.
4. $\log z = ax + (a + 1/a) \log y + b$. **5.** $z - a = bxe^{y/a}$.
6. $2\sqrt{az} = \log (x^a y) + b$. **7.** $\log z = ax + a^2y^2 + b$.
8. $3 \log z = 3a \log x - 2(a - y)^{3/2} + b$;
also (using Remark), $3 \log z = 3y(a + \log x) + (a + \log x)^3 + b$.
9. $2z = a^2e^{2x} + 2aye^x + b$.

SECTION XIII·9

1. $z = ax + by + \sqrt{a^2 + b^2 + 1}$, c.s., $x^2 + y^2 + z^2 = 1$, s.s.
2. $z = ax + by + a^2 + b^2$, c.s., $x^2 + y^2 + 4z = 0$, s.s.
3. $z = ax + by + 1/(a^2 + b^2)$, c.s., $27(x^2 + y^2) = 4z^3$, s.s.
4. $z = ax + by + a^2 + ab + b^2$, c.s., $x^2 - xy + y^2 + 3z = 0$, s.s.

SECTION XIII·10

1. $\phi(1/x - 1/y, 1/x - 1/z) = 0$. **2.** $z = cx^a y^{\sqrt{1-a^2}}$.
3. $3u = xyz + \phi(y/x, z/x)$.
4. $z = ax + by + ab$, c.s., $xy + z = 0$, s.s. **5.** $z = cx^a y^{1/a}$.
6. $\phi(x^3 - y^3, x^2 - z^2) = 0$. **7.** $z = a \log x + (a + a^2)y + b$.
8. $(a + 1)(z - b)^2 = 4(ax + y)$. **9.** $z = a\sqrt{x+y} + \sqrt{1-a^2}\sqrt{x-y} + b$.
10. $z^2 = a^2x^2 + (ay + b)^2$.
11. $z = a \log \sqrt{x^2 + y^2} + \sqrt{1 - a^2} \tan^{-1} (y/x) + b$.
12. $x^2 + y^2 + z^2 = z\phi(y/z)$. **13.** $(a - 1)(ax + y + b)z + 1 = 0$.
14. $z = axy + a^2(x + y) + b$. **15.** $z + \sqrt{x^2 + y^2 + z^2} = x^{1-n}\phi(y/x)$.
16. $2az = (ax + y)^2 + b$.
17. $(x - u)^3(x + y + z + u) = \phi\left(\frac{x - y}{z - u}, \frac{y - z}{z - u}\right)$.
18. $z = a \log x + (a^2 + 1) \log y + b$.
19. $3 \log z = 2(x + a)^{3/2} + 2(y - a)^{3/2} + b$.
20. $z^2 \sqrt{a^2 + 1} = 2axy + x^2 - y^2 + b$.
21. $z = a \log \sqrt{x^2 + y^2} + (1/a) \tan^{-1} (y/x) + b$.

---

* Attention should be called to the fact that no unique answer can be given for a complete solution. Other forms, just as good as the ones given here, may often be found by selecting various forms for ϕ.

22. $z^2(a - y^2) = (x + b)^2$. **23.** $x^2 + [z - y\phi(x)]^2 = 1$.

24. $3z = (x - a)^3 + 3a^2y + b$. **25.** $9x^{1/3}y^{1/3} - z = x\phi(y/x)$.

26. The family of planes $z = ax + \sqrt{k^2 - 1 - a^2}\,y + b$; for b a fixed constant, the corresponding planes envelop the cone

$$(k^2 - 1)(x^2 + y^2) - (z - b)^2 = 0,$$

whose vertex is the point $(0, 0, b)$.

27. The surfaces of revolution $z = \phi(x^2 + y^2)$.

28. The family of planes $z = ax + by + \sqrt{(k + c^2)a^2 + kb^2 + k}$, where the constant product is equal to k, and the fixed points are $(c, 0, 0)$ and $(-c, 0, 0)$. These envelop the quadric of revolution

$$x^2/(c^2 + k) + (y^2 + z^2)/k = 1,$$

which is an ellipsoid or an hyperboloid of two sheets, according as k is positive or negative.

29. The family of planes $z - ax - by = (kab)^{1/3}$. These envelop the surface $27xyz = k$.

30. The family of planes $z - ax - by = kab/(ab - a - b)$. These envelop the surface $x^{1/2} + y^{1/2} + z^{1/2} = k^{1/2}$.

SECTION XIV·2

2. $z = \phi(y + x) + \psi(y + 2x)$. **3.** $z = \phi_1(y) + \phi_2(y + 2x) + \phi_3(y + 5x)$.

4. $z = \phi_1(x) + \phi_2(y - x) + \phi_3(y + 3x)$.

SECTION XIV·3

1. $z = \phi(y + x) + x\psi(y + x)$.

2. $z = \phi_1(y + x) + \phi_2(y - x) + x\phi_3(y - x)$.

3. $z = \phi_1(x + y) + x\phi_2(x + y) + \phi_3(x - y) + x\phi_4(x - y)$.

SECTION XIV·5

4. $z = \phi(y - x) + \psi(3x + 2y) + \frac{1}{2}\sin(x + y) + (10x^4y + x^5)/80$.

5. $z = \phi(y - x) + \psi(y + 2x) - x^2y/2$.

SECTION XIV·6

3. $z = \phi(y - x) + e^{-2x}\psi(y + 2x) - \frac{1}{10}e^{2x+3y} - \frac{1}{6}\cos(2x + y)$.

4. $z = e^x\phi(y) + e^{-x}\psi(x + y) + \frac{1}{2}\sin(x + 2y) - xe^y$.

SECTION XIV·7

2. $z = \phi(y/x) + \psi(xy)$.

3. $z = \phi(x^2y) + x\psi(x^2y) + x^3y^4/30$.

4. $z = \phi(x^2 + y^2) + \psi(x^2 - y^2)$.

SECTION XIV·8

3. $u = Ae^{-ax} \sin ay,\ a > 0.$

4. $u = Ae^{-a(x+y)},\ a > 0;\ u = Ae^{a(y-x)},\ a > 0.$

SECTION XIV·9

4. (*a*) $f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$

(*b*) $f(x) = \frac{1}{\pi}\left(-\sin x + \frac{4 \sin 2x}{3} - \sin 3x + \frac{8 \sin 4x}{15} - \frac{\sin 5x}{3} + \cdots\right);$

$$b_n = \frac{2n}{\pi(n^2 - 1)} \text{ for } n \text{ even,}$$
$$= -\frac{1 - (-1)^{(n+1)/2}}{\pi(n + 1)} - \frac{1 - (-1)^{(n-1)/2}}{\pi(n - 1)} \text{ for } n \text{ odd.}$$

(*c*) $f(x) = \frac{4}{\pi}\left[\frac{2 \sin 2x}{3} + \frac{4 \sin 4x}{15} + \cdots + \frac{2n \sin 2nx}{(2n)^2 - 1} + \cdots\right].$

(*d*) $$f(x) = \frac{2}{\pi}\left[\frac{1 + e^{\pi}}{2} \sin x + \frac{2(1 - e^{\pi})}{5} \sin 2x + \cdots + \frac{n[1 + (-1)^{n-1}e^{\pi}]}{n^2 + 1} \sin nx + \cdots\right].$$

SECTION XIV·10

3. $$\theta = \frac{400}{\pi}\left[e^{-h^2\pi^2t/a^2} \sin \frac{\pi x}{a} + \tfrac{1}{3}e^{-9h^2\pi^2t/a^2} \sin \frac{3\pi x}{a} + \tfrac{1}{5}e^{-25h^2\pi^2t/a^2} \sin \frac{5\pi x}{a} + \cdots\right].$$

SECTION XIV·11

4. $x = \phi(z) + \psi(x + y + z).$ **5.** $x = \phi(y) + \psi(z).$

6. $y = \phi(ax + by + cz) + x\psi(ax + by + cz).$

SECTION XIV·12

3. $z = \phi(x) + y^2\psi(x) + xy^3/3.$

4. $z = \phi(x) + \psi(y) + x^2y^2/4.$

5. $z = \phi(y) + e^{-x}\psi(y) + x^2y/2 - xy.$

SECTION XIV·13

1. $z = \phi(x) + \psi(y) + \frac{1}{2}x^2 \log y + xy.$

2. $z = \phi(y + 3x) + \psi(y - 2x) + x^3y/6 + x^4/24.$

3. $z^2 = \phi(y) + x\psi(y) + x^3y^2.$

4. $z = \phi(x + y) + \psi(xy)$.
5. $z = \phi(y) + x^2\psi(y) + \frac{1}{2}x^2y \log x$.
6. $z = \phi(y + x) + e^{3x}\psi(y - x) - ye^{x+2y}$.
7. $z = \phi(xy) + x\psi(y/x) + (x - 1)y \log x$.
8. $z = x^n\phi(y/x) + x^{-n}\,\psi(y/x)$.
9. $z = \phi(y) + \psi(y) \log x + x^3y^2$.
10. $z = \phi(x) + \psi(x + y) + (x + y) \log x$.
11. $e^{-z} = \phi(x) + \psi(y)$.
12. $z = \phi(x + y) + x\psi(x + y) + \frac{1}{2}x^2(x + y)^2$.
13. $y + \phi(x + y + z) + x\psi(x + y + z) = 0$.
14. $y = \phi(x) + \psi(z)$.
15. $z = e^x\phi(y) + e^{-y}\psi(x) - xy + x - y + 1$.

ENERGY FLOW, BY MARKOWITZ

1 [illegible], of Biochemistry POTENTIAL BONDS

(2) Complexity is result of combinations

3 The information is structural

4 Energy is carried about in Energy rich phos bond

5 Ecology [SYSTEM]

(6) Cells

7 Universal Machine

8 Pop. produce mutants

9 DNA → m-RNA $\xrightarrow[\text{tRNA enzy}]{\text{Ribosomes}}$ Protein

10 Kin. of Reactions in Biology

11 Feed back used (or closed loop Control)

12 How Darwinism happens

New Frontiers

1 Submolecular Bio (after Chem)
(eg chem. gene → prod. protein)

(2) Quantitative Evolution

(3). Non ⇌ Thermodynamics

4 Engineering Optimization

Computer

ⓐ pop. fast
ⓑ invulnerable (constant)
ⓒ physical adaptability
ⓓ repair mechanism
ⓔ reliable – reassembly
ⓕ efficient – thermodynamically

equal

Limitation, irregularity; accidents,
Chance

Bio – Probability-wise
Quantum mech – still stat-wise
use very much to explain
certain Bio phenomena

James Lee
2511 Hearst Apt. 303 94709
848 0641